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The stability of unsteady cylinder flows

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First, the linear stability of the flow between two concentric cylinders when the outer one is at rest and the inner has angular velocity $\Omega\{1+\epsilon\cos\omega t\}$ is considered. In the limit in which ϵ and ω tend to zero it is found that the critical Taylor number at which instability first occurs is decreased by an amount of order ϵ^2 from its unmodulated value, the stabilizing effect at order $\epsilon^2\omega^2$ being slight. The limit in which ω tends to infinity with ϵ arbitrary is then studied. In this case it is found that the critical Taylor number is decreased by an amount of order $\epsilon^2\omega^{-3}$ from its unmodulated value.

Second, the effect of taking nonlinear terms into account is investigated. It is found that equilibrium perturbations of small but finite amplitude can exist under slightly supercritical conditions in both the high and low frequency limits. Some comparisons with experimental results are made, but these indicate that further theoretical work is needed for a broad band of values of ω. In appendix B it is shown how this can be done by an alternative formulation of the problem.

1. Introduction

Our concern is with the linear and nonlinear stability of the flow between concentric cylinders against a Taylor-vortex type of perturbation when the outer cylinder is at rest and the inner has angular velocity $\Omega\{1+\epsilon\cos\omega t\}$, t being the time and Ω, ω and ϵ constants. When ϵ is zero the appearance of Taylor vortices is predicted by linear stability theory when the Taylor number, which is proportional to Ω^2, reaches a certain critical value. The problem with ϵ non-zero has been investigated experimentally by Donnelly (1964), who found that the critical value of the Taylor number at which a Taylor-vortex type of flow appeared was increased from its unmodulated value. Moreover, he found that for all values of ϵ the maximum value of the critical Taylor number, i.e. the maximum enhancement of stability, always occurred at about the same value of the frequency ω. This value of ω corresponds to a value of 0·27 for a frequency parameter σ, which is defined later by (2.2) to be proportional to the ratio of the separation of the cylinders to the thickness of the Stokes layer associated with the oscillatory motion of the inner cylinder.

As a starting point we restrict attention to the stability of the basic unsteady flow to disturbances which are small enough for linearization to be a valid approximation. This is followed later by a nonlinear analysis. The procedure adopted is as follows.

In § 2 we determine the basic flow and obtain the partial differential equations governing the stability of this flow. These equations must be solved subject to there being no relative velocity at the walls of the cylinders. Following Venezian (1969) and Rosenblat & Herbert (1970), who considered the thermal analogue of the cylinder problem, we use the periodicity criterion to determine a 'boundary' between stability and instability. Venezian obtained a solution to the related Bénard convection problem by letting the parameter corresponding to ϵ be small and expanding in powers of this parameter. Rosenblat & Herbert used a WKB approach when the frequency of the basic temperature distribution was small. In this paper we consider in turn two other limiting types of procedure.

In § 3 we expand both the Taylor number T and the velocity field in terms of ϵ and σ. We seek a solution of the partial differential system governing the linear stability of the flow by letting ϵ tend to zero with σ/ϵ fixed and equal to α, say. This is done so that the dominant time dependences of the system 'balance' in a sense which we shall discuss later. A similar idea was used by DiPrima & Stuart (1972) in the mathematically related problem of the non-local stability of the flow between eccentric rotating cylinders. Thus we expand the perturbation velocity and the Taylor number in powers of ϵ and replace σ by $\alpha\epsilon$ in the partial differential system, equate like powers of ϵ and obtain ordinary differential systems in which the time variable appears only as a parameter. At order ϵ^0, the Taylor-vortex steady velocity field multiplied by an unknown function of time is obtained. A differential equation for this function follows from a solvability condition applied to the order-ϵ system. The order-ϵ term in the expansion of T is determined by the condition that the solution should be a periodic function of ωt. Higher-order terms in the expansion of T are obtained also. The first non-zero correction to T from its unmodulated value, which we denote by T_0, is of order ϵ^2.

In contrast to § 3, we consider in § 4 the limit of large σ with ϵ arbitrary. The time dependence of the basic flow is then confined to a thin layer near the inner cylinder, the Stokes layer; we shall refer to this layer as the inner layer. However, the interaction of the basic flow with the disturbance in this layer causes the disturbance velocity field to have a time dependence throughout the fluid. Hence a second Stokes layer, the 'outer' layer, is required at the outer cylinder to satisfy the boundary conditions. We shall refer to the region between the Stokes layers as the 'central' region. In each region we first expand the disturbance velocity field in a Fourier series in time, and then expand the Fourier coefficients and the Taylor number in powers of $\sigma^{-\frac{1}{2}}$. The disturbance velocity is then obtained in each region by equating like powers of $\sigma^{-\frac{1}{2}}$ and solving the resulting differential systems. The velocity is then matched in assumed domains of overlap. Concomitantly, terms in the expansion of T are determined by matching the steady parts of the velocity field. The analysis used in § 4 is related to that used by Schlichting (1932), Stuart (1966) and Riley (1967), who discussed the steady streaming induced by an oscillatory viscous flow. The first non-zero correction to T from T_0 is of order σ^{-3} and denoted by T_6/σ^3.

In § 5 we describe the numerical work required to solve the ordinary differential systems appearing in §§ 3 and 4. The results show that the first correction terms to T from T_0 in both limits are negative, thus suggesting a destabilization of the flow. However, higher-order terms in both expansions give a stabilizing effect. The discrepancy between these results and those of Donnelly (1964) leads us to ask whether nonlinear effects are important. This possibility is explored in the latter part of the paper.

In § 6, therefore, we return to the limit of ϵ tending to zero with σ/ϵ fixed, but include nonlinear terms, and obtain a solution to the differential system by the method of multiple scales. The Taylor number is again perturbed by an amount of order ϵ from its critical value T_0 for the steady problem. The perturbation velocity is also expanded in powers of ϵ and we find that the time-dependent amplitude A of the leading axially periodic Fourier mode satisfies the following differential equation:

$$\alpha \frac{dA}{d(\omega t)} = \frac{-\Gamma}{2T_0}\{T_1 + 2T_0 \cos \omega t\} A + a_1 A^3. \tag{1.1}$$

Here T_1 is the order-ϵ alteration to the perturbed Taylor number and Γ and a_1 are negative constants. Since a_1 is negative the A^3 term in (1.1) stabilizes the flow. Equation (1.1) has a solution which is periodic in ωt in which T_1 determines the elevation of the Taylor number above its unmodulated critical value.

In § 7 we examine the limit in which σ tends to infinity with ϵ arbitrary, assuming that the Taylor number T is given by

$$T = T_0 + T_6'/\sigma^3 + O(\sigma^{-\frac{7}{2}}). \tag{1.2}$$

It follows from the results of § 4 that the flow is stable to infinitesimally small disturbances if

$$T_6' < T_6.$$

To the order of magnitude in σ to which we work, we find that the nonlinear effects are only important through their effect on the steady part of the perturbation velocity. The amplitude A_s^0 of the leading steady Fourier mode is given by

$$A_s^{02} = \frac{\Gamma}{2a_1}\left\{\frac{T_6 - T_6'}{T_0}\right\}, \tag{1.3}$$

so that T_6 must be greater than T_6' if equilibrium perturbations are to exist.

In § 8 we discuss the relevance of this work to Donnelly's observations, and find that the low frequency nonlinear calculations explain some of his results but that our theory does not predict an optimum value of σ for the enhancement of stability. We also discuss other difficulties which arise when we try to compare the theoretical and experimental results.

2. The basic flow and the disturbance equations

We suppose that the viscous incompressible fluid between the infinitely long concentric cylinders of radii R_1 and R_2 has density ρ and kinematic viscosity ν. We assume throughout this paper that the separation d of the cylinders is small compared with R_1. Thus terms of order d/R_1 are neglected in the following analysis. We take cylindrical polar co-ordinates (r, θ, z), with the z axis along the axis of the cylinders, and take (u, v, w) as the corresponding velocity. If we define dimensionless variables ζ, ϕ and τ by

$$\zeta = (r - R_1)/d, \quad \phi = z/d, \quad \tau = \omega t,$$

then the velocity field between the cylinders when the outer one is at rest and the inner has angular velocity $\Omega\{1 + \epsilon \cos \omega t\}$ is $(0, \Omega R_1 \bar{V}(\zeta, \tau), 0)$, where

$$\bar{V} = \left\{1 - \zeta + \frac{\epsilon e^{i\tau} \sinh (i\sigma)^{\frac{1}{2}} (1-\zeta)}{\sinh (i\sigma)^{\frac{1}{2}}} + \text{complex conjugate}\right\}. \tag{2.1}$$

Here σ is the frequency parameter, mentioned in § 1, and is defined by

$$\sigma = \omega d^2/\nu. \tag{2.2}$$

Suppose that we perturb this flow such that the disturbed velocity field is $(u, v + \Omega R_1 \bar{V}, w)$. We rescale (u, v, w) by writing

$$u = (-\nu/2d)\, u^*, \quad v = \tfrac{1}{2}\Omega R_1 v^*, \quad w = (-\nu/2d)\, w^*,$$

and we can show from the momentum and continuity equations and the condition of zero relative velocity at the boundaries that u^*, v^* and w^* are determined by

$$\left.\begin{aligned} \left\{\mathscr{L} - \sigma \frac{\partial}{\partial \tau}\right\} \mathscr{L} u^* &= T \bar{V} \frac{\partial^2 v^*}{\partial \phi^2} - \frac{1}{2}\frac{\partial^2 Q_1}{\partial \phi^2} + \frac{1}{2}\frac{\partial^2 Q_2}{\partial \zeta\, \partial \phi}, \\ \left\{\mathscr{L} - \sigma \frac{\partial}{\partial \tau}\right\} v^* &= -\frac{\partial \bar{V}}{\partial \zeta} u^* - \tfrac{1}{2} Q_3, \\ \partial u^*/\partial \zeta + \partial w/\partial \phi &= 0, \\ u^* = v^* = w^* = 0, &\quad \zeta = 0, 1. \end{aligned}\right\} \tag{2.3}$$

In the above system we have introduced $\mathscr{L}$, Q_1, Q_2 and Q_3, which are defined by

$$\mathscr{L} \equiv \partial^2/\partial \zeta^2 + \partial^2/\partial \phi^2, \tag{2.4}$$

$$\left.\begin{aligned} Q_1 &= u^* \,\partial u^*/\partial \zeta + w^* \,\partial u^*/\partial \phi - \tfrac{1}{2} T v^{*2}, \\ Q_2 &= u^* \,\partial w^*/\partial \zeta + w^* \,\partial w^*/\partial \phi, \\ Q_3 &= u^* \,\partial v^*/\partial \zeta + w^* \,\partial v^*/\partial \phi. \end{aligned}\right\} \tag{2.5}$$

We have also defined the Taylor number T by

$$T = 2\Omega^2 R_1 d^3/\nu^2, \tag{2.6}$$

which represents the ratio of the destabilizing centrifugal and stabilizing viscous forces acting on the fluid. Finally we note that (2.3) are the so-called 'small gap' equations obtained from the full equations by letting d/R_1 tend to zero with ζ, τ, ϕ, u^*, v^*, w^*, T, σ and ϵ held fixed.

3. Linear stability for small frequencies

In this section we assume that the disturbance to the flow is small enough for linearization to be a valid procedure; thus we neglect the nonlinear terms Q_1, Q_2 and Q_3 which appear in (2.3). We further assume that the disturbance is periodic along the axis of the cylinder. Thus if a is a non-dimensional wavenumber we assume that u^* and v^* are proportional to $\cos a\phi$ and w^* is proportional to $\sin a\phi$. If we expand $\overline{V}$, given by (2.2), in (2.3) for small σ and drop the nonlinear terms and the star notation we can show that

$$\left.\begin{aligned}\{M-\sigma\,\partial/\partial\tau\}\,Mu &= -a^2Tv\{\chi_0+\epsilon\chi_1\cos\tau+\epsilon\sigma\chi_2\sin\tau\ldots\},\\ \{M-\sigma\,\partial/\partial\tau\}\,v &= u\{1+\epsilon\cos\tau+\epsilon\sigma\phi_2\sin\tau\ldots\},\\ u = v = \partial u/\partial\zeta &= 0, \quad \zeta = 0, 1,\end{aligned}\right\} \tag{3.1}$$

where
$$M \equiv \partial^2/\partial\zeta^2 - a^2, \tag{3.2a}$$

and u and v are now independent of ϕ. The first few functions $\chi_i(\zeta)$ appearing above are given by

$$\chi_0 = \chi_1 = 1-\zeta, \quad \chi_2 = \tfrac{1}{6}(\zeta^3-3\zeta^2+2\zeta), \tag{3.2b}$$

and for convenience we have defined $\phi_i = -d\chi_i/d\zeta$.

We now ask if it is possible to constrain ϵ and σ to tend to zero in such a way that the dominant τ dependences of the right- and left-hand sides of the two differential equations in (3.1) 'balance' in some sense. If we assume that T varies little from its unmodulated value T_0 we can see that the responses of the $\partial/\partial\tau$ terms on the left-hand sides of these equations are proportional to the τ dependences imposed by the $\epsilon\cos\tau$ terms on the right-hand sides if we have $\sigma \sim \epsilon$. Hence we write

$$\sigma = \alpha\epsilon \tag{3.3}$$

and let ϵ tend to zero with α fixed. The physical interpretation of this is that we are allowing the frequency and velocity amplitude of the oscillations of the inner cylinder to tend to zero in such a way that the oscillatory displacement $\alpha^{-1}(Td/2R_1)^{\frac{1}{2}}$ of the cylinder remains constant if T and d/R_1 are held fixed. We expand the perturbation velocities and the Taylor number in the form

$$u = u_0+\epsilon u_1+\epsilon^2 u_2\ldots, \tag{3.4a}$$

$$v = v_0+\epsilon v_1+\epsilon^2 v_2\ldots, \tag{3.4b}$$

$$T = T_0+\epsilon T_1+\epsilon^2 T_2\ldots. \tag{3.4c}$$

It is clear that, since changing ϵ to $-\epsilon$ does not change the physical problem under consideration, we should expect $T_i = 0$ for i odd, and this is found in the calculations. Substituting for σ, u, v and T from (3.3), (3.4) into (3.1) and equating terms of order ϵ^0 we obtain

$$\left.\begin{aligned}M^2u_0+a^2T_0\chi_0v_0 = u_0-Mv_0 &= 0,\\ u_0 = v_0 = \partial u_0/\partial\zeta = 0, \quad \zeta &= 0, 1.\end{aligned}\right\} \tag{3.5}$$

Since τ does not appear explicitly in (3.5) we have an ordinary differential equation whose solution is given by

$$(u_0, v_0) = B_0(\tau)\,(f_0(\zeta), g_0(\zeta)), \tag{3.6}$$

where f_0 and g_0 satisfy (3.5) with (u_0, v_0) replaced by (f_0, g_0) and the partial derivatives replaced by ordinary ones. The function $B_0(\tau)$ is determined by considering the order-ϵ system, which from (3.1), (3.3) and (3.4) is found to be

$$M^2u_1+a^2T_0\chi_0v_1 = \alpha\frac{dB_0}{d\tau}Nf_0-B_0\cos\tau a^2T_0\chi_1g_0-B_0a^2T_1\chi_0g_0, \qquad (3.7\,a)$$

$$u_1-Mv_1 = -\alpha\frac{dB_0}{d\tau}g_0-B_0\cos\tau f_0, \qquad (3.7\,b)$$

$$u_1 = v_1 = \partial u_1/\partial\zeta = 0, \quad \zeta = 0,1, \qquad (3.7\,c)$$

where

$$N \equiv d^2/d\zeta^2-a^2. \qquad (3.8)$$

After solving the order-ϵ^0 system, we have only B_0 and T_1 as unknowns on the right-hand sides of the differential equations in (3.7).

The function pair (f_0^+,g_0^+) adjoint to (f_0,g_0) satisfies the following differential system:

$$\left.\begin{aligned} N^2f_0^+ +g_0^+ = 0, \quad a^2T_0\chi_0f_0^+ -Ng_0^+ = 0,\\ f_0^+ = g_0^+ = df_0^+/d\zeta = 0, \quad \zeta = 0,1. \end{aligned}\right\} \qquad (3.9)$$

The eigenvalues a and T_0 of (3.5) and (3.9) are identical but, as the form of the equations shows, (f_0,g_0) and (f_0^+,g_0^+) are not the same.

Having introduced (f_0^+,g_0^+) we can show that the condition that (3.7) has a solution is that the integral from $\zeta = 0$ to $\zeta = 1$ of the sum of f_0^+ times the right-hand side of (3.7 a) and g_0^+ times the right-hand side of (3.7 b) is zero. (See, for example, Ince 1927, p. 213.) Thus we have

$$\alpha\frac{dB_0}{d\tau}\left\{\int_0^1[f_0^+Nf_0-g_0^+g_0]\,d\zeta\right\}-B_0\cos\tau\left\{\int_0^1[a^2T_0\chi_1f_0^+g_0+g_0^+f_0]\,d\zeta\right\} = B_0a^2T_1\int_0^1\chi_1f_0^+g_0\,d\zeta, \qquad (3.10)$$

which is an ordinary differential equation for B_0 having a solution periodic in τ if $T_1 = 0$. The function B_0 is then of the form

$$B_0(\tau) = A\exp\{-(\Gamma/\alpha)\sin\tau\}, \qquad (3.11)$$

where the constant Γ is given by

$$\Gamma = \int_0^1[a^2T_0\chi_1f_0^+g_0+g_0^+f_0]\,d\zeta\Big/\int_0^1[g_0^+g_0-f_0^+Nf_0]\,d\zeta \qquad (3.12)$$

and A is a constant, dependent on the parameters of the problem, which can only be determined by considering the corresponding nonlinear problem.

Having determined B_0 we can see from (3.7) that (u_1,v_1) is of the form

$$(u_1,v_1) = B_0(\tau)\cos\tau(f_1(\zeta),g_1(\zeta))+B_1(\tau)\,(f_0(\zeta),g_0(\zeta)). \qquad (3.13)$$

The function pair (f_1,g_1) satisfies the following system:

$$\left.\begin{aligned} N^2f_1+a^2T_0\chi_0g_1 = -\Gamma Nf_0-a^2T_0\chi_1g_0,\\ f_1-Ng_1 = \Gamma g_0-f_0,\\ f_1 = g_1 = dg_1/d\zeta = 0, \quad \zeta = 0,1. \end{aligned}\right\} \qquad (3.14)$$

The solvability condition at order ϵ^2 gives an ordinary differential equation for

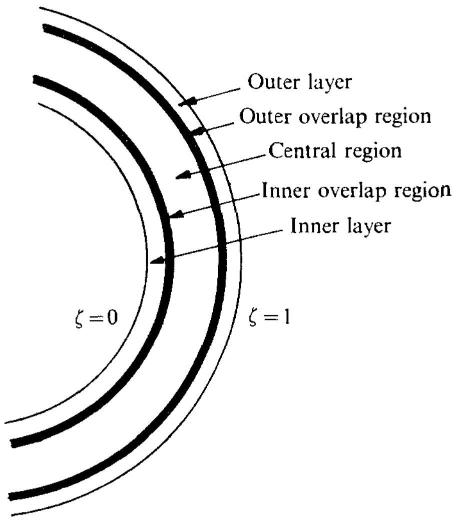

FIGURE 1. The different regions in the disturbance velocity field for large σ.

the function $B_1(\tau)$ appearing in (3.13). The solution of this equation is periodic in τ if

$$a^2T_2 = -\frac{1}{2}\frac{\int_0^1 [f_0^+\{Nf_1+a^2T_0\chi_1 g_0\}+g_0^+\{f_1-\Gamma g_1\}]\,d\zeta}{\int_0^1 \chi_0 f_0^+ g_0\,d\zeta} \tag{3.15}$$

Further terms in the expansion of T can be obtained by considering the higher-order differential systems. It suffices here to say that, up to order ϵ^4, we can write T in the form

$$T = T_0+\epsilon^2T_2+\epsilon^4[\alpha^2T_{40}+T_{42}]+O(\epsilon^6), \tag{3.16}$$

where T_{40} and T_{42} are independent of α and are determined from tedious integral conditions which are to be found in the author's thesis (1973).

4. Linear theory for large frequencies

We now investigate the limit in which σ tends to infinity with ϵ arbitrary, in which case the Stokes layer associated with the oscillatory motion of the inner cylinder is thin compared with the gap between the cylinders. If we let σ tend to infinity in (2.1) we can show that

$$\bar{V} \sim 1-\zeta+\tfrac{1}{2}\epsilon\{\exp[-(i\sigma)^{\frac{1}{2}}\zeta+i\tau]+\exp[(-i\sigma)^{\frac{1}{2}}-i\tau]\}, \tag{4.1}$$

so that the time-dependent part of the basic flow decays exponentially to zero when ζ becomes of order $\sigma^{-\frac{1}{2}}$. In contrast to this behaviour we shall see that the disturbance velocity field is time dependent throughout the fluid. Hence, as well as having a Stokes layer at the inner cylinder, the disturbance velocity field must also have a Stokes layer at the outer cylinder in order to satisfy the no-slip boundary condition there. As stated earlier, we refer to these layers as the 'inner' and 'outer' layers respectively, and the region between these layers will be called the 'central' region. (See figure 1.)

We first introduce the following new variables:

$$\zeta^* = 1-\zeta, \quad \eta^* = \zeta^*(\tfrac{1}{2}\sigma)^{\frac{1}{2}}, \quad \eta = \zeta(\tfrac{1}{2}\sigma)^{\frac{1}{2}}. \tag{4.2 a, b, c}$$

Thus η^* and η are Stokes-layer variables for the 'outer' and 'inner' layers respectively. We define (u, v), (U, V) and (u^*, v^*) to be the disturbance velocities in the inner layer, central region and outer layer respectively. We again assume that the flow is periodic along the z axis, so that u, v, U, etc. are all proportional to $\cos a\phi$, a again being a non-dimensional wavenumber. If we neglect the non-linear terms in (2.3), i.e. Q_1, Q_2 and Q_3, we can use (4.1) and (4.2) to show that the appropriate differential equations to determine the above pairs are

$$\left\{\frac{\partial}{\partial\eta^2}-\frac{2a^2}{\sigma}-\frac{2\partial}{\partial\tau}\right\}\left\{\frac{\partial^2}{\partial\eta^2}-\frac{2a^2}{\sigma}\right\}u = -\frac{4a^2Tv}{\sigma^2}\left\{1-\eta\left(\frac{2}{\sigma}\right)^{\frac{1}{2}} + \frac{\epsilon}{2}[\exp[-\eta(1+i)+i\tau]+\text{c.c.}]\right\}, \tag{4.3 a}$$

$$\left\{\frac{\partial^2}{\partial\eta^2}-\frac{2a^2}{\sigma}-\frac{2\partial}{\partial\tau}\right\}v = \frac{2u}{\sigma}\left\{1+\frac{\epsilon\sigma^{\frac{1}{2}}}{2^{\frac{3}{2}}}[(1+i)\exp[-\eta(1+i)+i\tau]+\text{c.c.}]\right\}, \tag{4.3 b}$$

$$\{M-\sigma\,\partial/\partial\tau\}MU + a^2T\chi_0 V = 0, \tag{4.4 a}$$

$$\{M-\sigma\,\partial/\partial\tau\}V - U = 0, \tag{4.4 b}$$

$$\left\{\frac{\partial^2}{\partial\eta^{*2}}-\frac{2a^2}{\sigma}-2\frac{\partial}{\partial\tau}\right\}\left\{\frac{\partial^2}{\partial\eta^{*2}}-\frac{2a^2}{\sigma}\right\}u^* = -\frac{4a^2T2^{\frac{1}{2}}\eta^*v^*}{\sigma^{\frac{5}{2}}}, \tag{4.5 a}$$

$$\left\{\frac{\partial^2}{\partial\eta^{*2}}-\frac{2a^2}{\sigma}-2\frac{\partial}{\partial\tau}\right\}v^* = \frac{2u^*}{\sigma}, \tag{4.5 b}$$

where c.c. denotes 'complex conjugate' and M is defined by (3.2 a). We can also see from (2.3) that the required boundary conditions are

$$u = v = \partial u/\partial\eta = 0, \quad \eta = 0, \tag{4.6 a}$$

$$u^* = v^* = \partial u^*/\partial\eta^* = 0, \quad \eta^* = 0, \tag{4.6 b}$$

and we stipulate that the perturbation velocities must match in the assumed regions of overlap.

We now expand the perturbation velocities in each region in Fourier series in time. This is possible since we are seeking solutions periodic in τ. Thus we write

$$u = u_s + \frac{1}{2}\sum_{n=1}^{\infty}\{u_n e^{in\tau} + \tilde{u}_n e^{-in\tau}\}, \tag{4.7}$$

$$U = U_s + \frac{1}{2}\sum_{n=1}^{\infty}\{U_n e^{in\tau} + \tilde{U}_n e^{-in\tau}\}, \tag{4.8}$$

$$u^* = U_s + \frac{1}{2}\sum_{n=1}^{\infty}\{u_n^* e^{in\tau} + \tilde{u}_n^* e^{-in\tau}\}, \tag{4.9}$$

and note that the azimuthal velocity components can be expanded in a similar manner. In (4.7)–(4.9) a tilde denotes a complex conjugate. The expression

(u_s, v_s) represents the steady part of the disturbance velocity in the 'inner' layer. In both the other two layers we denote the steady part of the disturbance velocity by the same expression (U_s, V_s). This is possible because there is no behaviour of Stokes-layer type for the steady part of the disturbance velocity in the 'outer' layer. However, in the 'inner' layer the interaction of the basic flow and the disturbance causes the steady part of the disturbance velocity to have terms proportional to exponentially decaying terms. Thus it is necessary to distinguish between the steady parts of the disturbance velocity in the 'inner' layer and away from it.

Suppose that we substitute the Fourier expansions of u, v, U, etc., into (4.3)–(4.5) and equate like powers of $e^{in\tau}$, for $n = 0, 1, 2, \ldots$; then we find that the equations for the central and outer functions (U_s, V_s), (U_n, V_n) and (u_n^*, v_n^*), for $n = 1, 2, \ldots$, are not coupled. However, in the inner layer this is not the case and we find that

$$\left\{D^2 - \frac{2a^2}{\sigma}\right\}^2 u_s = -\frac{4a^2T}{\sigma^2}\left\{v_s\left[1-\eta\left(\frac{2}{\sigma}\right)^{\frac{1}{2}}\right] + \frac{\epsilon}{4}(\tilde{v}_1 e^{-\eta(1+i)} + \text{c.c.})\right\}, \quad (4.10\,a)$$

$$\left\{D^2 - \frac{2a^2}{\sigma}\right\} v_s = \frac{2}{\sigma}\left\{u_s + \frac{\epsilon\sigma^{\frac{1}{2}}}{2^{\frac{5}{2}}}(\tilde{u}_1(1+i)e^{-\eta(1+i)} + \text{c.c.})\right\}, \quad (4.10\,b)$$

$$\left\{D^2 - \frac{2a^2}{\sigma} - 2i\right\}\left\{D^2 - \frac{2a^2}{\sigma}\right\} u_1 = -\frac{4a^2T}{\sigma^2}\left\{v_1\left[1-\eta\left(\frac{2}{\sigma}\right)^{\frac{1}{2}}\right] + \epsilon\left(v_s e^{-\eta(1+i)} + \frac{v_2}{2}e^{-\eta(1-i)}\right)\right\}, \quad (4.11\,a)$$

$$\left\{D^2 - \frac{2a^2}{\sigma} - 2i\right\} v_1 = \frac{2}{\sigma}\left\{u_1 + \epsilon\left(\frac{\sigma}{2}\right)^{\frac{1}{2}}\left(u_s(1+i)e^{-\eta(1+i)} + \frac{u_2}{2}(1-i)e^{-\eta(1-i)}\right)\right\}, \quad (4.11\,b)$$

and for $n \geqslant 2$

$$\left\{D^2 - \frac{2a^2}{\sigma} - 2in\right\}\left\{D^2 - \frac{2a^2}{\sigma}\right\} u_n = -\frac{4a^2T}{\sigma^2}\left\{v_n\left[1-\eta\left(\frac{2}{\sigma}\right)^{\frac{1}{2}}\right] + \frac{\epsilon}{2}(v_{n-1}e^{-\eta(1+i)} + v_{n+1}e^{-\eta(1-i)})\right\}, \quad (4.12\,a)$$

$$\left\{D^2 - \frac{2a^2}{\sigma} - 2in\right\} v_n = \frac{2}{\sigma}\left\{u_n + \frac{\epsilon}{2}\left(\frac{\sigma}{2}\right)^{\frac{1}{2}}(u_{n-1}(1+i)e^{-\eta(1+i)} + u_{n+1}(1-i)e^{-\eta(1-i)})\right\}. \quad (4.12\,b)$$

The coupling of these equations arises from the terms proportional to $e^{-\eta(1\pm i)}$ on the right-hand sides of (4.3). We recall that, in the absence of modulation, the Taylor-vortex flow was given by (f_0, g_0), this function pair being determined by (3.5). We can show from (3.5) that near $\zeta = 0$

$$f_0 \sim \zeta^2 \sim \sigma^{-1}, \quad g_0 \sim \zeta \sim \sigma^{-\frac{1}{2}},$$

and since we are again seeking a solution which is in some sense a perturbation from the steady problem with $\epsilon = 0$ we assume that the correct scaling for (u_s, v_s) follows from the above. Hence we have

$$u_s \sim \sigma^{-1}, \quad v_s \sim \sigma^{-\frac{1}{2}},$$

and this scaling for (u_s, v_s), together with (4.10)–(4.12), suggests the following scalings for (u_1, v_1), (u_2, v_2), etc.:

$$u_{2n-1} \sim \sigma^{-\frac{5}{2}n}, \quad u_{2n} \sim \sigma^{-\frac{1}{2}(5n+2)}, \quad v_{2n-1} \sim \sigma^{-\frac{1}{2}(5n-2)}, \quad v_{2n} \sim \sigma^{-\frac{1}{2}(5n+1)} \qquad (4.13\,a\text{–}d)$$

for $n = 1, 2, \ldots$. Hence we expand the above functions as follows:

$$u_s = \sigma^{-1}\{u_s^0 + u_s^1\sigma^{-\frac{1}{2}} + \ldots\}, \quad v_s = \sigma^{-\frac{1}{2}}\{v_s^0 + v_s^1\sigma^{-\frac{1}{2}} + \ldots\}, \qquad (4.14\,a, b)$$

$$u_1 = \sigma^{-\frac{5}{2}}\{u_1^0 + u_1^1\sigma^{-\frac{1}{2}} + \ldots\}, \quad v_1 = \sigma^{-\frac{3}{2}}\{v_1^0 + v_1^1\sigma^{-\frac{1}{2}} + \ldots\}, \qquad (4.14\,c, d)$$

etc. The Taylor number is expanded in the form

$$T = T_0 + T_1\sigma^{-\frac{1}{2}} + \ldots. \qquad (4.15)$$

If the above expansions are substituted into (4.10)–(4.12) it is found that the first five terms in the expansion of (u_s, v_s) can be determined without any knowledge of (u_1, v_1), (u_2, v_2), etc. Having determined these terms we find that we can then calculate the first five terms in the expansion of (u_1, v_1). These terms enable us to calculate the next five terms in the expansion of (u_s, v_s) and the first five terms in the expansion of (u_2, v_2). Continuing in this way we can determine any number of terms in the expansions of (u_s, v_s), (u_1, v_1), (u_2, v_2), etc. The essential details of this rather tedious process are given in appendix A. The important fact which emerges from this process is that (u_1, v_1), (u_2, v_2), etc., contain terms which are not exponentially small at the edge of the inner layer and it is for this reason that, unlike the basic flow, the disturbance velocity field is time dependent everywhere.

In the central region we expand (U_n, V_n) in the form

$$U_n = \mu_n(\sigma)\{U_n^0 + U_n^1\sigma^{-\frac{1}{2}} + \ldots\}, \quad V_n = \mu_n(\sigma)\{V_n^0 + V_n^1\sigma^{-\frac{1}{2}} + \ldots\}. \quad (4.16\,a, b)$$

The functions of σ, namely $\mu_n(\sigma)$, appearing above are determined by the matching conditions in the region where the inner layer and the central region overlap. In the outer layer we expand the functions u_n^* and v_n^* in a similar manner to U_n and V_n but with $\mu_n(\sigma)$ replaced by another function of σ, namely $\nu_n(\sigma)$, which is similarly determined by applying matching conditions, but this time in the region where the central region and outer layer overlap. The matching conditions in each overlap region also determine certain constants which arise in the functions u_n^0, etc. Of particular interest are μ_1 and ν_1, which determine the orders of magnitude of the dominant unsteady velocity away from the inner layer. We find that

$$\mu_1 = \sigma^{-\frac{5}{2}}, \quad \nu_1 = \sigma^{-3}. \qquad (4.17\,a, b)$$

The functions μ_1 and ν_1 are determined by the conditions that U_1 and u_1 are of the same order in σ in the inner overlap region and that U_1 and u_1^* are of the same order in σ in the outer overlap region respectively.

We now turn to the steady part of the perturbation velocity away from the inner layer, and this we expand in the form

$$U_s = \{U_s^0 + U_s^1\sigma^{-\frac{1}{2}} + \ldots\}, \quad V_s = \{V_s^0 + V_s^1\sigma^{-\frac{1}{2}} + \ldots\}. \qquad (4.18\,a, b)$$

Here we have assumed that U_s and V_s are of order σ^0; this would otherwise be found later by applying the matching conditions. We can use (4.4)–(4.9) and (4.18) to show that the U_s^i and V_s^i are determined by

$$\left.\begin{aligned} N^2U_s^0+a^2T_0\chi_0 V_s^0 &= 0, \\ U_s^0-NV_s^0 &= 0, \\ U_s^0 = V_s^0 = dU_s^0/d\zeta = 0, \quad \zeta &= 1, \end{aligned}\right\} \tag{4.19}$$

and

$$\left.\begin{aligned} N^2U_s^i+a^2T_0\chi_0 V_s^i &= -a^2\chi_0\sum_{r=0}^{i-1} T_{i-r}V_s^r \\ U_s^i-NV_s^i &= 0 \\ U_s^i = V_s^i = dU_s^i/d\zeta = 0, \quad \zeta &= 1, \end{aligned}\right\} \quad \text{for} \quad i \geqslant 1. \tag{4.20}$$

The matching condition where the inner layer and the central region overlap is found to be that in the inner overlap region

$$\begin{aligned} U_s^i \sim{} & \sum_{i=0}^{7}\sigma^{-\frac{1}{2}i}\{S_1(A_i, B_i, C_i, a, T_0, \zeta)+\text{terms proportional to } T_k \ (0<k\leqslant i)\} \\ & +\sigma^{-3}\left\{\frac{\alpha_1}{2^{\frac{1}{2}}}\left[\zeta-\frac{a^4\zeta^5}{120}\right]-\frac{a^2T_0\gamma_1\zeta^4}{24}+\frac{a^2\gamma_1\zeta^5}{120}+O(\zeta^6)\right\} \\ & +\sigma^{-\frac{7}{2}}\left\{\frac{\alpha_2}{2^{\frac{1}{2}}}\left(\zeta-\frac{a^4\zeta^5}{120}\right)-\frac{a^2T_0\gamma_2\zeta^4}{24}+\frac{a^2\gamma_2\zeta^5}{120}+\beta_1\left(1-\frac{a^4\zeta^4}{24}\right)+O(\zeta^6)\right\}+O(\sigma^{-4}), \end{aligned} \tag{4.21 a}$$

$$\begin{aligned} V_s^i \sim{} & \sum_{i=0}^{7}\sigma^{-\frac{1}{2}i}\{S_2(A_i, B_i, C_i, a, T_0, \zeta)+\text{terms proportional to } T_k \ (0<k\leqslant i)\} \\ & +\sigma^{-3}\left\{\frac{\alpha_1}{6\times 2^{\frac{1}{2}}}\left[\zeta^3+\frac{a^2\zeta^5}{20}\right]+\gamma_1\left[1+\frac{a^2\zeta^2}{2}+\frac{a^4\zeta^4}{24}\right]+O(\zeta^6)\right\} \\ & +\sigma^{-\frac{7}{2}}\left\{\frac{\alpha_2}{6\times 2^{\frac{1}{2}}}\left[\zeta^3+\frac{a^2\zeta^5}{20}\right]+\gamma_2\left[1+\frac{a^2\zeta^2}{2}+\frac{a^4\zeta^4}{24}\right]\right. \\ & \left.+\beta_1\left[\frac{\zeta^2}{2}+\frac{a^2\zeta^4}{24}\right]+O(\zeta^6)\right\}+O(\sigma^{-4}), \end{aligned} \tag{4.21 b}$$

where

$$\alpha_1 = \frac{13a^2\epsilon^2B_0T_02^{\frac{1}{2}}}{32}, \quad \alpha_2 = \frac{B_1}{B_0}\alpha_1+\frac{231a^2\epsilon^2A_0T_02^{\frac{1}{2}}}{256}, \tag{4.22 a, b}$$

$$\beta_1 = \frac{-25a^2\epsilon^2B_0T_02^{\frac{1}{2}}}{64}, \tag{4.22 c}$$

$$\gamma_1 = \frac{-a^2\epsilon^2C_0T_02^{\frac{1}{2}}}{16}, \quad \gamma_2 = \frac{C_1\gamma_1}{C_0}+\frac{5a^3\epsilon^2C_0T_0\coth a}{32}. \tag{4.22 d, e}$$

Moreover S_1 and S_2 represent the following series:

$$S_1 = \frac{A_i}{2^{\frac{3}{2}}}\left\{\zeta^3+\frac{a^2\zeta^5}{10}\right\}+\frac{B_i}{2}\left\{\zeta^3+\frac{a^2\zeta^4}{6}\right\}-\frac{a^2C_i\zeta^5}{120\times 2^{\frac{1}{2}}}+O(\zeta^6), \tag{4.23 a}$$

$$S_2 = \frac{A_i\zeta^5}{40\times 2^{\frac{1}{2}}}+\frac{B_i\zeta^4}{24}+\frac{C_i}{2^{\frac{1}{2}}}\left\{\zeta+\frac{a^2\zeta^3}{6}+\frac{a^4\zeta^5}{120}\right\}+O(\zeta^6), \tag{4.23 b}$$

where A_i, B_i and C_i are constants to be determined shortly. (For more details of the derivation of (4.21) the reader should consult the appendix.) We first note that (f_0, g_0), defined by (3.5), has the following form for small ζ:

$$(f_0, g_0) = (S_1(A, B, C, a, T_0, \zeta), S_2(A, B, C, a, T_0, \zeta)), \tag{4.24}$$

where A, B and C are given by

$$A = \tfrac{1}{3} \times 2^{\frac{1}{2}} f_0'''(0), \quad B = f_0''(0), \quad C = 2^{\frac{1}{2}} g_0'(0). \tag{4.25a,b,c}$$

A prime denotes a derivative with respect to ζ, and S_1 and S_2 are defined by (4.23). Hence if we choose $(U_s^0, V_s^0) = (f_0, g_0)$, (4.19) is automatically satisfied and if we put $A_0 = A$, $B = B_0$ and $C = C_0$, then U_s^0 and V_s^0 are of the form required by (4.21), where the central region and inner layer overlap. Similarly if we put

$$T_i = 0, \quad (U_s^i, V_s^i) = (C_i/C_0)(f_0, g_0), \quad i = 1, 2, \ldots, 5,$$

then the next five terms (U_s^i, V_s^i) satisfy (4.20) and are of the form required by (4.18) and (4.21) where the central region and the inner layer overlap. With the above choices for T_1, T_2, etc., we can show that (U_s^6, V_s^6) is determined by

$$N^2 U_s^6 + a^2 T_0 \chi_0 V_s^6 = -a^2 T_6 \chi_0 g_0, \tag{4.26a}$$

$$U_s^6 - N V_s^6 = 0, \tag{4.26b}$$

with boundary conditions

$$U_s^6 = V_s^6 = dU_s^6/d\zeta = 0, \quad \zeta = 1. \tag{4.27}$$

In addition, from (4.18) and (4.21) we require that in the inner overlap region

$$U_s^6 \sim S_1(A_6, B_6, C_6, a, T_0, \zeta) + \frac{\alpha_1}{2^{\frac{1}{2}}}\left(\zeta - \frac{a^4\zeta^5}{120}\right) - \frac{a^2 T_0 \gamma_1 \zeta^4}{24} + \frac{a^2\gamma_1\zeta^5}{120} - \frac{a^2 C_0 T_6 \zeta^5}{120 \times 2^{\frac{1}{2}}} + O(\zeta)^6, \tag{4.28a}$$

$$V_s^6 \sim S_2(A_6, B_6, C_6, a, T_0, \zeta) + \frac{\alpha_1}{6 \times 2^{\frac{1}{2}}}\left(\zeta^3 + \frac{a^2\zeta^5}{20}\right) + \gamma_1\left(1 + \frac{a^2\zeta^2}{2} + \frac{a^4\zeta^4}{24}\right) + O(\zeta^6). \tag{4.28b}$$

The above series are just the small-ζ series solutions of (4.26) with boundary conditions

$$U_s^6 = 0, \quad V_s^6 = \gamma_1, \quad dU_s^6/d\zeta = \alpha_1/2^{\frac{1}{2}}, \quad \zeta = 0. \tag{4.29}$$

Therefore, if we consider (4.26) with boundary conditions (4.27) and (4.29), the solution will automatically satisfy the requirements on (U_s^6, V_s^6) away from the 'inner' layer and for some A_6, B_6 and C_6 will satisfy (4.28). Thus the problem reduces to solving (4.26) subject to (4.27) and (4.29). In fact, since we are only interested in finding T_6, we merely use the condition that this system has a solution; this gives

$$a^2 T_6 = \frac{\gamma_1 g_0^{+\prime}(0) + \alpha_1 2^{-\frac{1}{2}} f_0^{+\prime\prime}(0)}{\int_0^1 \chi_0 f_0^+ g_0 \, d\zeta},$$

where (f_0^+, g_0^+) is the adjoint function pair defined by (3.10). Using (4.22) and (4.25), we can show that the above expression can be written in the form

$$T_6 = \epsilon^2 T_0 \frac{13 f_0'''(0) f_0^{+''}(0) - 4 g_0'(0)\, g_0^{+'}(0)}{32 \int_0^1 \chi_0 f_0^+ g_0 \, d\zeta}, \tag{4.30}$$

and a similar procedure for the order-$\sigma^{-\frac{7}{2}}$ terms in (4.18) and (4.21) shows that

$$T_7 = \epsilon^2 T_0 \frac{40a \coth a g_0'(0)\, g_0^{+'}(0) + 100 f_0''(0) f_0^{+'''}(0) + 77 f_0''''(0) f_0^{+''}(0)}{128 \times 2^{\frac{1}{2}} \int_0^1 \chi_0 f_0^+ g_0 \, d\zeta}. \tag{4.31}$$

Then T_8, T_9, etc. can be obtained by a similar method if more terms in the expansions of the perturbation velocities are evaluated. However, we have seen that, to order σ^{-4}, T may be written in the form

$$T = T_0 + T_6 \sigma^{-3} + T_7 \sigma^{-\frac{7}{2}} + \ldots, \tag{4.32}$$

where T_6 and T_7 are determined by (4.30) and (4.31) respectively and T_0 is the Taylor number for the steady problem with $\epsilon = 0$.

5. The numerical work

If we wish to obtain the critical number T_c associated with (3.16), we must take into account the variation of a with ϵ near its critical value for the problem with $\epsilon = 0$. A calculation similar to the one given by Venezian (1969) shows that, if this effect is taken into account, T_c is given by

$$T_c = T_0^c + \epsilon^2 T_2^c + \epsilon^4 \left\{ \alpha^2 T_{40}^c + T_{42}^c - \frac{\frac{1}{2}(\partial T_2^c/\partial a)^2}{\partial^2 T_0^c/\partial a^2} \right\} + O(\epsilon^6), \tag{5.1}$$

where T_0^c, etc., denote T_0, etc., evaluated with a equal to a^c, its critical value for the problem with $\epsilon = 0$. Similarly the critical Taylor number associated with (4.32) is given by

$$T_c = T_0^c + T_6^c/\sigma^3 + T_7^c/\sigma^{\frac{7}{2}} + O(\sigma^{-4}). \tag{5.2}$$

All the computations were for the critical case, and as a starting point, we assumed the following well-known values of a^c and T_0^c:

$$a^c = 3{\cdot}1266, \quad T_0^c = 3389{\cdot}9. \tag{5.3}$$

The ordinary differential systems arising in §§ 3 and 4 were solved by a Runge–Kutta scheme with 40 steps following the method outlined by Eagles (1971). The solutions for (f_0, g_0) and (f_0^+, g_0^+) were in good agreement with those of DiPrima & Stuart (1972), when normalized in the same way. The computations showed that the constant Γ, defined by (3.12), has the numerical value $-26{\cdot}18$ and that (5.1) and (5.2) are expressible in the form

$$T_c = 3389{\cdot}9 - 208{\cdot}6\epsilon^2 + 1{\cdot}7\epsilon^2\sigma^2 + O(\epsilon^4, \epsilon^2\sigma^4), \tag{5.4}$$

$$T_c = 3389{\cdot}9 \left\{ 1 - \frac{\epsilon^2}{\sigma^3} \times 10^4 \left\{ 4{\cdot}898 - \frac{84{\cdot}81}{\sigma^{\frac{1}{2}}} \right\} \right\} + O(\sigma^{-4}). \tag{5.5}$$

In (5.4) we have replaced α by σ/ϵ. We see immediately that the first and second correction terms in (5.4) and (5.5) produce destabilizing and stabilizing effects respectively.

These formulae do not overlap in σ, and do not give the pronounced stabilization found by Donnelly (1964). One possible remedy is to discuss the nonlinear aspects, and this we do in the next section.

6. Nonlinear theory for small frequencies

If we replace $\bar{V}$ in (2.3) by its asymptotic form for small σ and drop the star notation we obtain

$$\left\{\mathscr{L}-\sigma\frac{\partial}{\partial\tau}\right\}\mathscr{L}u = T\{\chi_0+\epsilon\chi_1\cos\tau+\epsilon\sigma\chi_2\sin\tau+\dots\}\frac{\partial^2 v}{\partial\phi^2}-\frac{1}{2}\frac{\partial^2 Q_1}{\partial\phi^2}+\frac{1}{2}\frac{\partial^2 Q_2}{\partial\phi\,\partial\zeta}, \quad (6.1a)$$

$$\{\mathscr{L}-\sigma\,\partial/\partial\tau\}v = \{1+\epsilon\cos\tau+\epsilon\sigma\phi_2\sin\tau+\dots\}u-\tfrac{1}{2}Q_3, \quad (6.1b)$$

$$\partial u/\partial\zeta+\partial w/\partial\phi = 0, \quad (6.1c)$$

where the functions χ_i and ϕ_i are defined by (3.2b). The relevant boundary conditions are

$$u = v = w = 0, \quad \zeta = 0, 1. \quad (6.2)$$

Following the method of § 3 we seek a solution of the above partial differential system by letting ϵ tend to zero with σ/ϵ fixed and equal to α say. We expand u, v, w and T in the form

$$T = T_0+\epsilon T_1+\dots, \quad (6.3a)$$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \epsilon^{\frac{1}{2}}\begin{pmatrix} u_{01}\cos a\phi \\ v_{01}\cos a\phi \\ w_{01}\sin a\phi \end{pmatrix} + \epsilon\left[\begin{pmatrix} u_{10} \\ v_{10} \\ w_{10} \end{pmatrix} + \begin{pmatrix} u_{12}\cos 2a\phi \\ v_{12}\cos 2a\phi \\ w_{12}\sin a\phi \end{pmatrix}\right]$$

$$+\epsilon^{\frac{3}{2}}\left[\begin{pmatrix} u_{21}\cos a\phi \\ v_{21}\cos a\phi \\ w_{21}\sin a\phi \end{pmatrix} + \begin{pmatrix} u_{23}\cos 3a\phi \\ v_{23}\cos 3a\phi \\ w_{23}\sin 3a\phi \end{pmatrix}\right] + O(\epsilon^2), \quad (6.3b)$$

where

$$\mathbf{u}_{01} = \begin{pmatrix} u_{01} \\ v_{01} \\ w_{01} \end{pmatrix}, \quad \text{etc.}$$

are independent of ϕ and a is again a non-dimensional wavenumber. This expansion procedure is the same as the one used by DiPrima & Stuart (1973), who considered the stability of the flow between eccentric rotating cylinders. Indeed much of the analysis of this section is related to their work. The $\epsilon^{\frac{1}{2}}$ scaling factor in (6.3b) follows from (6.3a) and the usual argument that, if T is slightly greater than T_0, the amplitude of the disturbance is then proportional to $(T-T_0)^{\frac{1}{2}}$.

We define L_p by

$$L_p \equiv \begin{pmatrix} (\partial^2/\partial\zeta^2-p^2a^2)^2 & p^2a^2T_0\chi_0 & 0 \\ 1 & -\partial^2/\partial\zeta^2+p^2a^2 & 0 \\ \partial/\partial\zeta & 0 & pa \end{pmatrix},$$

where p is a non-negative integer and L_p^* is defined to be L_p with the partial derivatives replaced by ordinary derivatives. If we replace σ by $\alpha\epsilon$ in (6.1), substitute for u, v, w and T from the above into (6.1) and (6.2) and equate terms of order $\epsilon^{\frac{1}{2}}$, we obtain

$$L_1 \mathbf{u}_{01} = 0, \tag{6.4a}$$

$$\mathbf{u}_{01} = 0, \quad \zeta = 0, 1. \tag{6.4b}$$

We can write the solution of this system in the form

$$\mathbf{u}_{01} = A(\tau)\begin{pmatrix} f_0(\zeta) \\ g_0(\zeta) \\ -a^{-1}\,df_0(\zeta)/d\zeta \end{pmatrix}, \tag{6.5a, b, c}$$

where f_0 and g_0 are defined by (3.5). If we now substitute for u, v, w and T from (6.3) into (6.1) and (6.2), equate terms of order ϵ and use (6.5), we obtain ordinary differential systems for $\mathbf{u}_{10}$ and $\mathbf{u}_{12}$ whose solutions may be written in the form

$$\mathbf{u}_{10} = A^2\begin{pmatrix} 0 \\ g_2(\zeta) \\ 0 \end{pmatrix}, \quad \mathbf{u}_{12} = A^2\begin{pmatrix} f_3(\zeta) \\ g_3(\zeta) \\ h_3(\zeta) \end{pmatrix}, \tag{6.6), (6.7}$$

where g_2, f_3, g_3 and h_3 are

$$L_0^*\begin{pmatrix} 0 \\ g_2 \\ 0 \end{pmatrix} = -\frac{1}{4}\frac{d}{d\zeta}\begin{pmatrix} 0 \\ f_0 g_0 \\ 0 \end{pmatrix}, \tag{6.8a}$$

$$g_2 = 0, \quad \zeta = 0, 1, \tag{6.8b}$$

and

$$L_2^*\begin{pmatrix} f_3 \\ g_3 \\ h_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -a^2 T_0 g_0^2 + \dfrac{df_0}{d\zeta}\dfrac{d^2 f_0}{d\zeta^2} - f_0\dfrac{d^3 f_0}{d\zeta^3} \\ -\tfrac{1}{2} g_0^2 \dfrac{d}{d\zeta}\left(\dfrac{f_0}{g_0}\right) \\ 0 \end{pmatrix}, \tag{6.9a}$$

$$f_3 = g_3 = h_3 = 0, \quad \zeta = 0, 1. \tag{6.9b}$$

Substituting for u, v, w and T from (6.3) into (6.1) and (6.2), equating terms of order $\epsilon^{\frac{3}{2}}$ and using (6.5)–(6.7), we find that $\mathbf{u}_{21}$ is determined by

$$L_1 \mathbf{u}_{21} = \begin{pmatrix} \alpha(dA/d\tau)\,N f_0 - A a^2 T_0 \chi_1 g_0 \cos\tau - A a^2 T_1 \chi_0 g_0 + A^3 F_1(\zeta) \\ -\alpha(dA/d\tau)\,g_0 - A f_0 \cos\tau + A^3 G_1(\zeta) \\ 0 \end{pmatrix}, \tag{6.10a}$$

$$\mathbf{u}_{21} = 0, \quad \zeta = 0, 1, \tag{6.10b}$$

where the functions $F_1(\zeta)$ and $G_1(\zeta)$ are defined by

$$F_1 = -\frac{1}{8}\left\{f_0\frac{d^3 f_3}{d\zeta^3} + 2\frac{df_0}{d\zeta}\frac{d^2 f_3}{d\zeta^2} - \frac{d^2 f_0}{d\zeta^2}\frac{df_3}{d\zeta} - 2\frac{d^3 f_0}{d\zeta^3} f_3\right\} + \frac{3a^2}{8}\left\{f_0\frac{df_3}{d\zeta} + 2\frac{df_0}{d\zeta} f_3\right\} - \frac{a^2 T_0}{8}\{g_0 g_3 + 2 g_0 g_2\}, \tag{6.11a}$$

$$G_1 = \frac{1}{4}\left\{f_0\frac{dg_3}{d\zeta} + f_3\frac{dg_0}{d\zeta} + 2 g_3\frac{df_0}{d\zeta} + \frac{g_0}{2}\frac{df_3}{d\zeta} + 2 f_0\frac{dg_2}{d\zeta}\right\}, \tag{6.11b}$$

and the operator N, arising from (3.12), is defined by (3.8). When $p = 1$ a differential system of the type (6.10) only has a solution when a certain orthogonality condition is satisfied. In this case we can show that the required condition yields an equation for $A(\tau)$, namely

$$\alpha\, dA/d\tau = -\Gamma\{\cos\tau + T_1/2T_0\}A + a_1 A^3, \qquad (6.12)$$

where Γ is given by (3.12) and has the numerical value $-26{\cdot}18$. The constant a_1 is defined by

$$a_1 = \int_0^1 \{f_0^+ F_1 + g_0^+ G_1\} d\zeta \Big/ \int_0^1 \{g_0^+ g_0 - f_0^+ N f_0\} d\zeta, \qquad (6.13)$$

where (f_0^+, g_0^+) is the adjoint function pair defined by (3.9), and F_1 and G_1 are as defined by (6.11). The constant a_1 is in fact related to the constant $\bar{a}_1$ introduced by Davey (1962) as follows: if we choose the function pair (f_0, g_0) equal to the function pair $(\bar{u}_1, v_1)$ of Davey's work, then

$$a_1 = \tfrac{1}{8}\bar{a}_1,$$

while the functions F_1, $\bar{u}_2$ and v_2 introduced by Davey are given by

$$F_1 = -4g_2, \quad \bar{v}_2 = -4g_3, \quad \bar{u}_2 = -4f_3.$$

Davey's numerical work shows that, with $g_0(\tfrac{1}{2}) = 1$, a_1 has the numerical value $-10{\cdot}05$.

Finally, suppose that the outer cylinder is at rest and that the inner one has angular velocity $\Omega_1\{1 + \epsilon f(\omega t)\}$. Then, for small σ, the dimensionless velocity

$$\bar{V} = \left\{ \sum_{i=0}^{\infty} \left[\epsilon \sigma^i \chi_{i+1}(\zeta) \frac{d^i f}{d\tau^i} \right] + \chi_0 \right\},$$

and the method described above leads to the following equation for the corresponding amplitude function $A(\tau)$:

$$\alpha\, dA/d\tau = -\Gamma\{f(\tau) + T_1/2T_0\} + a_1 A^3. \qquad (6.14)$$

Here a_1 and Γ are as defined earlier.

It has been pointed out to the author by one of the referees that an alternative approach to the problem is as follows. For any given value of T_1 and any function $f(\tau)$ of τ we can define $f^*(\tau)$ by

$$f^*(\tau) = f + T_1/2T_0, \qquad (6.15)$$

and then (6.14) can be written in the form

$$\alpha\, dA/d\tau = -\Gamma f^*(\tau) A + a_1 A^3. \qquad (6.16)$$

This equation is the one which would be obtained by setting $T_1 = 0$ initially and absorbing any order-ϵ correction to T from T_0 into the function $f(\tau)$. However, our approach is more helpful in that if we set $f(\tau) = 0$ the method reduces to the usual type of stability analysis. Also our approach is similar to that used by DiPrima & Stuart (1973), so that using this approach enables us to see the equivalence of the problems more easily.

Equation (6.14) is a 'Bernouilli' type equation and, if we use the usual substitution for such equations and use A^{-2} as a variable, we can show that

$$[A^{-2}\Phi(x)]_0^\tau = -\frac{2a_1}{\alpha}\int_0^\tau \Phi(x)\,dx, \tag{6.17}$$

where Φ is defined by

$$\Phi = \exp -\frac{\Gamma}{\alpha}\left(2\int^x f(g)\,dy + \frac{T_1 x}{T_0}\right). \tag{6.18}$$

As a special case we put $f(\tau) = \tanh\tau$, in which case the speed of the inner cylinder changes slowly from $\Omega(1-\epsilon)$ at $\tau = -\infty$ to $\Omega(1+\epsilon)$ at $\tau = +\infty$. We can see from (6.14) that, as we would expect, this equation admits an equilibrium amplitude solution as $\tau \to \infty$ which is just the equilibrium amplitude solution for the steady problem with the Taylor number based on the final speed of the inner cylinder.

We return now to the special case $f(\tau) = \cos\tau$ and note that (6.17) contains an unknown constant $A(0)$. We now invoke the condition that A should be a periodic function of τ. This determines $A(0)$ and we can then write $A(\tau)$ in the form

$$A^{-2}(\tau) = -\frac{2a_1}{\alpha}\left\{\frac{\int_0^{2\pi}\Psi(x)\,dx + [\Psi(2\pi)-1]\int_0^\tau \Psi(x)\,dx}{[\Psi(2\pi)-1]\,\Psi(\tau)}\right\}, \tag{6.19}$$

where $$\Psi(x) = \exp\frac{-\Gamma}{\alpha}\left\{2\sin x + \frac{T_1 x}{T_0}\right\}.$$

In general this form cannot be simplified further; we can however consider a second limit, with respect to T_1/T_0 or α. Thus, in the limit in which T_1/T_0 tends to infinity with α fixed, we can use (6.19) to show that

$$A(\tau) \sim (\Gamma T_1/2a_1 T_0)^{\frac{1}{2}}\{1 + O(T_1/T_0)^{-1}\}. \tag{6.20}$$

The dominant term on the right-hand side of (6.20) corresponds to the equilibrium amplitude solution for the unmodulated problem with the same Taylor number. Thus as the flow becomes more and more supercritical the effect of modulation becomes unimportant.

In a similar manner we can show by taking the second limit $\alpha \to \infty$ with T_1/T_0 fixed that

$$A(\tau) \sim (\Gamma T_1/2a_1 T_0)^{\frac{1}{2}}\{1 + O(\alpha^{-1})\}. \tag{6.21}$$

The dominant terms on the right-hand sides of (6.20) and (6.21) are the same so that we conclude that for ϵ and σ small but σ/ϵ large modulation has a negligible effect on the equilibrium amplitude.

Suppose now that we consider the limits of small T_1/T_0 and α. We investigate the result of taking the second limit $T_1/T_0 \to 0$ and then the further (third) limit $\alpha \to 0$. If we let T_1/T_0 tend to zero in (6.19) with α held fixed we obtain

$$A^{-2}(\tau) \sim \frac{a_1 T_0}{\pi\Gamma T_1}\exp\left(\frac{2\Gamma\sin\tau}{\alpha}\right) I_0\left(-\frac{2\Gamma}{\alpha}\right)\left\{1 + O\left(\frac{T_1}{T_0}\right)\right\}, \tag{6.22}$$

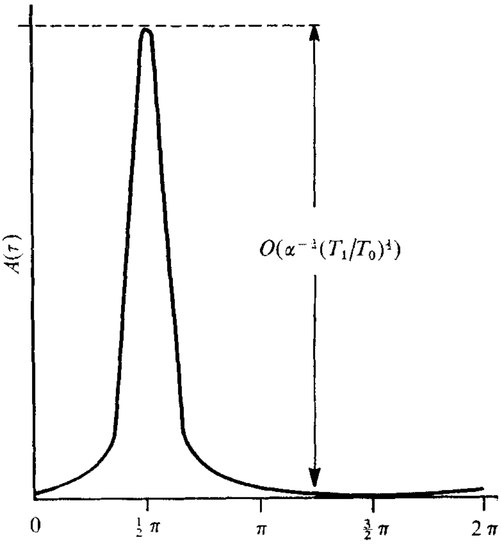

FIGURE 2. Amplitude as a function of τ for small α and T_1/T_0.

where I_0 is a modified Bessel function of zero order. If we now let α tend to zero in (6.22) we obtain

$$A(\tau) \sim \left[\frac{-T_1\pi^{\frac{3}{2}}(-2\Gamma)^{\frac{1}{2}}}{a_1 T_0}\right]^{\frac{1}{2}} \alpha^{-\frac{1}{4}} \exp\frac{\Gamma}{\alpha}(1-\sin\tau)\left[1+O\left(\frac{T_1}{T_0},\alpha\right)\right]. \qquad (6.23)$$

The τ dependence of $A(\tau)$ is then similar to that of $B_0(\tau)$ obtained in the linear flow frequency theory of § 3. We can see from (6.23) that $A(\tau)$ then has its maximum value at $\tau = \frac{1}{2}\pi$. It is also of interest to note that the average torque on the inner cylinder associated with the dominant term in (6.23) is independent of α. We show A as a function of τ for small α and T_1/T_0 in figure 2.

7. Nonlinear theory for large frequencies

We now investigate the possibility of the existence of equilibrium perturbations of small but finite size in the limit in which σ tends to infinity but with ϵ arbitrary. We consider only the case when the inner cylinder has angular velocity $\Omega_1\{1+\epsilon\cos\omega t\}$ and so we can seek periodic solutions from the outset. Thus we can Fourier analyse in ωt and use the method of § 4.

We recall that, in the linear theory of § 4, the effect of modulation in the 'central' region first appears at order σ^{-3} in the expansion of the steady component of the perturbation velocity in powers of $\sigma^{-\frac{1}{2}}$, when the dominant term is of order σ^0. Hence we perturb the Taylor number in the form

$$T = T_0 + T_6'/\sigma^3 + O(\sigma^{-\frac{7}{2}}), \qquad (7.1)$$

in order that the effects of modulation and nonlinearities appear at the same order when we expand in powers of $\sigma^{-\frac{1}{2}}$. It is important at this stage to distinguish between T_6', given in (7.1), and T_6, introduced in § 4. We recall that T_6 is the coefficient of the order-σ^{-3} term in the expansion of the critical Taylor number in powers of $\sigma^{-\frac{1}{2}}$, whereas T_6' is determined by (7.1) for any given value of T. The flow is stable or unstable according to linear theory depending on whether T_6' is less or greater than T_6.

As stated earlier in § 6 we know that without modulation the Taylor-vortex velocity field is of order $(T-T_0)^{\frac{1}{2}}$ when T is slightly greater than T_0. Thus we expect that the dominant steady fundamental velocity in the central region should be of order $\sigma^{-\frac{3}{2}}$. The linear theory of § 4 then leads us to the following expansion for u, v and w in the central region:

$$u = \sigma^{-\frac{3}{2}}\left\{U_s^0+\frac{U_s^1}{\sigma^{\frac{1}{2}}}+\ldots+\frac{e^{i\tau}}{\sigma^{\frac{5}{2}}}\left[U_1^0+\frac{U_1^1}{\sigma^{\frac{1}{2}}}+\ldots\right]\right.$$
$$\left.+\,\text{c.c.}+\frac{e^{2i\tau}}{\sigma^{\frac{7}{2}}}\left[U_2^0+\frac{U_2^1}{\sigma^{\frac{1}{2}}}+\ldots\right]+\text{c.c.}+O(\sigma^{-5})\right\}\cos a\phi, \quad (7.2\,a)$$

$$v = \sigma^{-\frac{3}{2}}\left\{V_s^0+\frac{V_s^1}{\sigma^{\frac{1}{2}}}+\ldots+\frac{e^{i\tau}}{\sigma^{\frac{7}{2}}}\left[V_1^0+\frac{V_1^1}{\sigma^{\frac{1}{2}}}+\ldots\right]\right.$$
$$\left.+\,\text{c.c.}+\frac{e^{2i\tau}}{\sigma^{\frac{9}{2}}}\left[V_2^0+\frac{V_2^1}{\sigma^{\frac{1}{2}}}+\ldots\right]+\text{c.c.}+O(\sigma^{-6})\right\}\cos a\phi, \quad (7.2\,b)$$

$$w = \sigma^{-\frac{3}{2}}\left\{W_s^0+\frac{W_s^1}{\sigma^{\frac{1}{2}}}+\ldots+\frac{e^{i\tau}}{\sigma^{\frac{5}{2}}}\left[W_1^0+\frac{W_1^1}{\sigma^{\frac{1}{2}}}+\ldots\right]\right.$$
$$\left.+\,\text{c.c.}+\frac{e^{2i\tau}}{\sigma^{\frac{7}{2}}}\left[W_2^0+\frac{W_2^1}{\sigma^{\frac{1}{2}}}+\ldots\right]+\text{c.c.}+O(\sigma^{-5})\right\}\sin a\phi, \quad (7.2\,c)$$

where c.c. denotes 'complex conjugate'. However, if we let σ tend to infinity we see from (2.1) that, away from the inner layer,

$$\bar{V} \sim 1-\zeta. \quad (7.3)$$

Expansions (7.2) are clearly no longer suitable if we wish to retain the nonlinear terms Q_1, Q_2 and Q_3 in (2.3). In order to take these nonlinear effects into account we modify these expansions to give

$$u = \sigma^{-\frac{3}{2}}\left\{U_s^0+\frac{U_s^1}{\sigma^{\frac{1}{2}}}+\ldots+\frac{e^{i\tau}}{\sigma^{\frac{5}{2}}}\left[U_1^0+\frac{U_1^1}{\sigma^{\frac{1}{2}}}+\ldots\right]+\text{c.c.}+\frac{e^{2i\tau}}{\sigma^{\frac{7}{2}}}\left[U_2^0+\frac{U_2^1}{\sigma^{\frac{1}{2}}}+\ldots\right]\right.$$
$$\left.+\,\text{c.c.}+O(\sigma^{-5})\right\}\cos a\phi+\sigma^{-3}\left\{U_s^{00}+\frac{U_s^{10}}{\sigma^{\frac{1}{2}}}+\ldots\right.$$
$$\left.+\left[U_s^{02}+\frac{U_s^{12}}{\sigma^{\frac{1}{2}}}+\ldots\right]\cos 2a\phi\right\}+O(\sigma^{-\frac{11}{2}}), \quad (7.4\,a)$$

$$v = \sigma^{-\frac{3}{2}}\left\{V_s^0+\frac{V_s^1}{\sigma^{\frac{1}{2}}}+\ldots+\frac{e^{i\tau}}{\sigma^{\frac{7}{2}}}\left[V_1^0+\frac{V_1^1}{\sigma^{\frac{1}{2}}}+\ldots\right]+\text{c.c.}+\frac{e^{2i\tau}}{\sigma^{\frac{9}{2}}}\left[V_2^0+\frac{V_2^1}{\sigma^{\frac{1}{2}}}+\ldots\right]\right.$$
$$\left.+\,\text{c.c.}+O(\sigma^{-6})\right\}\cos a\phi+\sigma^{-3}\left\{V^{00}+\frac{V_s^{10}}{\sigma^{\frac{1}{2}}}+\ldots\right.$$
$$\left.+\left[V_s^{02}+\frac{V_s^{12}}{\sigma^{\frac{1}{2}}}+\ldots\right]\cos 2a\phi\right\}+O(\sigma^{-\frac{11}{2}}), \quad (7.4\,b)$$

$$w = \sigma^{-\frac{3}{2}}\left\{W_s^0 + \frac{W_s^1}{\sigma^{\frac{1}{2}}} + \ldots + \frac{e^{i\tau}}{\sigma^{\frac{5}{2}}}\left[W_1^0 + \frac{W_1^1}{\sigma^{\frac{1}{2}}} + \ldots\right] + \text{c.c.} + \frac{e^{2i\tau}}{\sigma^{\frac{7}{2}}}\left[W_2^0 + \frac{W_2^1}{\sigma^{\frac{1}{2}}} + \ldots\right]\right.$$
$$\left. + \text{c.c.} + O(\sigma^{-5})\right\}\sin a\phi + \sigma^{-3}\left\{W_s^{00} + \frac{W_s^{10}}{\sigma^{\frac{1}{2}}} + \ldots\right.$$
$$\left. + \left[W_s^{02} + \frac{W_s^{12}}{\sigma^{\frac{1}{2}}} + \ldots\right]\sin 2a\phi\right\} + O(\sigma^{-\frac{11}{2}}), \quad (7.4\,c)$$

where the terms in these expansions with three indices are produced by nonlinear interactions.

From now on we shall use the words fundamental, mean, first harmonic, etc. with reference to the ϕ dependence only. The nonlinear interaction of the steady fundamental components of velocity with themselves leads to the steady mean and first-harmonic terms of order σ^{-3} in the above expansions. The nonlinear interactions involving the unsteady fundamental terms produce steady and unsteady mean and first-harmonic terms which are at most of order $\sigma^{-\frac{11}{2}}$. Since we shall consider terms only up to order $\sigma^{-\frac{9}{2}}$, these terms are negligible for our purposes. The dominant steady mean and first-harmonic terms produced by the interaction described above interact nonlinearly with the dominant steady fundamental terms to produce steady fundamental terms of order $\sigma^{-\frac{9}{2}}$. Similar terms are produced by the nonlinear interaction of the order-$\sigma^{-\frac{11}{2}}$ terms with themselves and the other terms in the above expansion, but these terms will be at most of order $\sigma^{-\frac{19}{2}}$, and so negligible. Thus we see that in the 'central' region the steady fundamental terms up to order $\sigma^{-\frac{9}{2}}$ are unaffected by any nonlinear interactions involving unsteady terms.

We recall that in the high frequency linear theory of § 4 the steady part of the perturbation velocity away from the inner layer had no Stokes-layer type of dependence, i.e. contained no exponentially decaying terms in the outer layer. This was in contrast to the steady component in the inner layer, which contained exponentially decaying terms caused by the nonlinear interaction of the time-dependent parts of the basic flow and perturbation velocity in this layer. A similar nonlinear interaction between the unsteady parts of the perturbation velocity with themselves in the outer layer leads to similar terms in the steady perturbation velocity in the outer layer if the terms Q_1, Q_2 and Q_3 in (2.3) are retained. Hence we must distinguish between the steady fundamental components in the 'central' region and the 'outer' layer. However, we can show that in both the 'inner' and 'outer' layers the residual steady fundamental components of velocity at the edges of these layers are first affected by nonlinearities at order $\sigma^{-\frac{9}{2}}$ (when the dominant steady fundamental component is of order $\sigma^{-\frac{3}{2}}$), and the effect is independent of the nonlinear interaction of the unsteady parts of the perturbation velocity. Thus the first-order nonlinear correction to the linear theory of § 4 is independent of the time dependence of the basic flow.

Having said this the solution of the problem becomes trivial since all the information which we require is embedded in §§ 4 and 6. If we substitute for u, v and w from (7.4) into (2.3) (with $\overline{V} = 1-\zeta$) and equate steady fundamental terms of order $\sigma^{-\frac{3}{2}}$ we find that the vector (U_s^0, V_s^0, W_s^0) satisfies the differential equations in (6.4). The matching conditions in the inner and outer overlap regions require that (U_s^0, V_s^0, W_s^0) there match onto the small-ζ and small-$(1-\zeta)$ series solutions of these equations with the boundary conditions that this vector vanishes at $\zeta = 0, 1$ respectively. Thus we can write

$$(U_s^0, V_s^0, W_s^0) = A_s^0(f_0, g_0, -a^{-1}df_0/d\zeta), \tag{7.5}$$

where (f_0, g_0) satisfies (3.5) and A_s^0 is an unknown amplitude constant to be determined. In a similar manner we can show that for $n = 1, 2, \ldots, 5$

$$(U_s^n, V_s^n, W_s^n) = A_s^n(f_0, g_0, -a^{-1}df_0/d\zeta),$$

and similarly we can show by considering the mean and first-harmonic terms of order σ^{-3} that

$$(U_s^{00}, V_s^{00}, W_s^{00}) = (A_s^0)^2(0, g_2, 0), \tag{7.6a}$$

$$(U_s^{02}, V_s^{02}, W_s^{02}) = (A_s^0)^2(f_3, g_3, h_3), \tag{7.6b}$$

where the functions g_2, f_3, g_3 and h_3 are determined by (6.8) and (6.9). If we substitute for u, v and w from (7.4) into (2.3) and equate steady fundamental terms of order $\sigma^{-\frac{9}{2}}$ we obtain

$$\left.\begin{aligned} N^2U_s^9 + a^2T_0\chi_0 V_s^9 &= -A_s^0a^2T_6'\chi_0 g_0 + (A_s^0)^3F_1(\zeta), \\ U_s^9 - NV_s^9 &= (A_s^0)^3G_1(\zeta), \end{aligned}\right\} \tag{7.7}$$

where F_1 and G_1 are given by (6.11). The matching condition in the outer overlap region requires that (U_s^9, V_s^9) there matches onto the small-$(1-\zeta)$ series solution of (7.7) with boundary conditions

$$U_s^9 = V_s^9 = dU_s^9/d\zeta = 0, \quad \zeta = 1. \tag{7.8}$$

Without modulation the corresponding condition in the inner overlap region would be that (U_s^9, V_s^9) there matches onto the small-ζ series solution with the conditions (7.8) at $\zeta = 0$. However, with modulation the nonlinear interaction of the basic flow and the disturbance in the inner layer affects the matching conditions in a similar way to that observed earlier in § 4.

An analysis shows that, if modulation is taken into account, the boundary conditions at $\zeta = 0$ are the conditions appropriate for (U_s^6, V_s^6) in the high frequency linear theory of § 4. Thus we require that

$$U_s^9 = 0, \quad V_s^9 = -\frac{a^2\epsilon^2A_s^0}{8}\frac{dg_0}{d\zeta}, \quad \frac{dU_s^9}{d\zeta} = \frac{13a^2\epsilon^2T_0A_s^0}{32}\frac{d^2f_0}{d\zeta^2}, \quad \zeta = 0, \tag{7.9}$$

and so U_s^9 and V_s^9 are given by the solution of (7.7) with boundary conditions (7.8) and (7.9). The condition that this system has a solution reduces to

$$a^2T_6'\int_0^1 \chi_0 f_0^+ g_0 d\zeta - (A_s^0)^2\int_0^1\{f_0^+F_1 + g_0^+G_1\}d\zeta = \frac{a^2\epsilon^2T_0}{32}\left[13\frac{d^2f_0^+}{d\zeta^2}\frac{d^2f_0}{d\zeta^2} - 4\frac{dg_0^+}{d\zeta}\frac{dg_0}{d\zeta}\right]_{\zeta=0},$$

where (f_0^+, g_0^+) is the adjoint function pair defined by (3.10), and $\chi_0 = 1-\zeta$. The

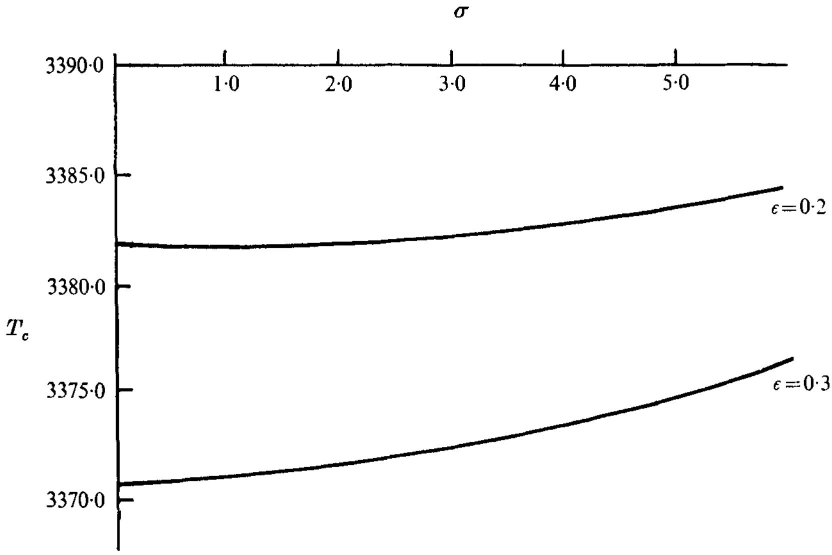

FIGURE 3. The critical Taylor number as a function of σ in the low frequency limit.

terms in this equation are more recognizable after a few substitutions. We can use (3.12), (4.30) and (6.13) to show that

$$A_s^0 = \left\{\frac{\Gamma}{2a_1}\left(\frac{T_6' - T_6}{T_0}\right)\right\}^{\frac{1}{2}}, \tag{7.10}$$

where a_1 is given by (6.13) and Γ and T_6 are given by (3.12) and (4.30). If (7.10) is to have a real solution we require $T_6' \geqslant T_6$, so that (7.1) indicates that finite amplitude perturbations can exist only when T is greater than its critical value in the linear theory of § 4.

8. Discussion of results

We have seen that the critical Taylor number at which instability first occurs in the limit in which ϵ and σ tend to zero is given by (5.5). Thus the dominant correction to T_c from its unmodulated value is negative, suggesting destabilization. However, for fixed ϵ, T_c increases as σ increases from zero and this is consistent with Donnelly's work. In figure 3 we show the variation of T_c with σ for fixed values of ϵ. We have also calculated the terms of order ϵ^4 and $\epsilon^2\sigma^4$ in (5.5). The former term is about -1300, whilst the latter is zero to two decimal places.

In the limit in which σ tends to infinity T_c is given by (5.6). We again see that the first and second correction terms suggest destabilization and stabilization respectively. Thus in both limits the first correction term, in contrast to the second, is not consistent with Donnelly's results. We show the variation of T_c with $\sigma^{\frac{1}{2}}$ for fixed ϵ in figure 4. It should be said that, although modulation is usually thought of as having a stabilizing effect, other mechanical systems can be made

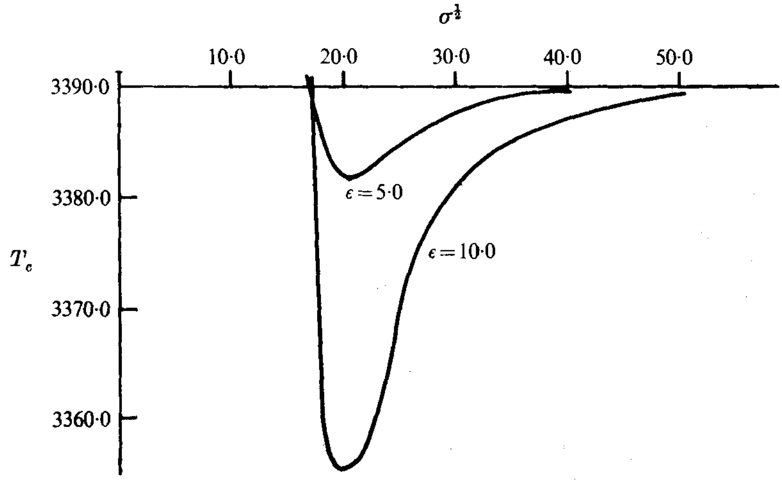

FIGURE 4. The critical Taylor number as a function of $\sigma^{\frac{1}{2}}$ in the high frequency limit.

less stable by modulation. For example a simple pendulum hanging vertically can be made unstable by suitably oscillating its support. (See Corben & Stehle 1960, p. 67.) Also Benjamin & Ursell (1954) have shown that oscillating a vessel containing fluid can cause the fluid surface to become unstable.

We should also comment that the methods used in §§ 3 and 4 can also be used for the related Bénard convection problem considered by Venezian (1969) and Rosenblat & Herbert (1970). Our results for the low frequency problem differ from those given by Venezian (1969). However, Dr Herbert, at Imperial College, in some unpublished work using a Galerkin method, obtained our result, thus giving reinforcement to the present work. For details of this problem the reader should consult Hall (1973).

In view of the disagreement of linear theory and experiment we must discuss the experimental work of Donnelly (1964) in more detail. As stated earlier, he considered the flow between concentric cylinders when the outer one is at rest and the inner one has angular velocity $\Omega\{1+\epsilon\cos\omega t\}$. Before going further, we first summarize the important features of the stability of the unmodulated problem.

When the outer cylinder is at rest and the inner one has angular velocity Ω_1, it can be shown that the flow first becomes unstable when the Taylor number reaches the value 3389·9. For T slightly greater than this value the nonlinear theory of Davey (1962) shows that equilibrium perturbations to the flow can exist. The amplitude of the velocity of the Taylor-vortex flow is then proportional to $(T-T_0)^{\frac{1}{2}}$. It can also be shown that such equilibrium amplitude flows cannot exist for T less than T_0. Thus for T less than T_0 the amplitude of the Taylor-vortex flow is zero, and when T reaches the value T_0 the amplitude begins to grow like $(T-T_0)^{\frac{1}{2}}$.

With this in mind, Donnelly defined the critical Taylor number to be that value for which a slight increase in Ω caused the amplitude of the Taylor-vortex flow to

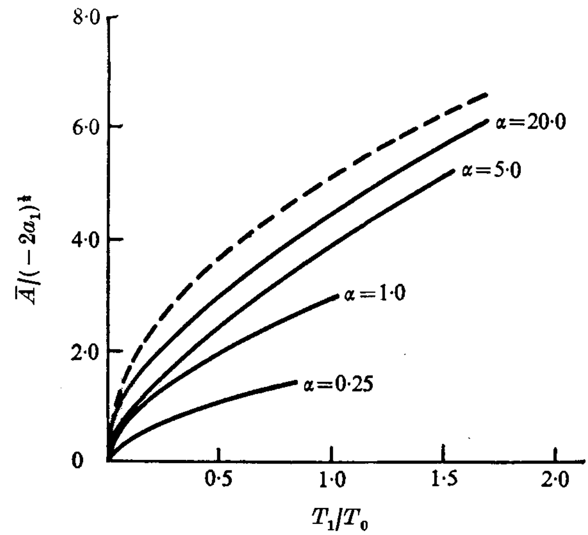

FIGURE 5. $\bar{A}$ as a function of T_1/T_0. ---, value of $\bar{A}$ for the unmodulated problem with the same Taylor number.

increase rapidly. With this definition of the critical Taylor number he found that the flow was stabilized for all ϵ and σ in the sense that the critical Taylor number was always greater than T_0, the maximum enhancement for all ϵ occurring when the parameter σ took the value 0·27.

We now see whether the low frequency nonlinear results can explain the discrepancy between theory and experiment. The first difficulty which we must overcome is to decide which property of the time-dependent amplitude was actually measured by Donnelly in his experiments. A relevant property of A might be its mean value $\bar{A}$ defined by

$$\bar{A} = \frac{1}{2\pi}\int_0^{2\pi} A\, d\tau. \tag{8.1}$$

We note from § 6 that A is known only in integral form. Thus A and $\bar{A}$ must be evaluated by an integration routine. In figure 5 we show $\bar{A}$ as a function of T_1/T_0 for different values of α, in comparison with the corresponding equilibrium amplitude for the unmodulated flow at the same Taylor number. We see that, as suggested by (6.23), the effect of modulation vanishes as α tends to infinity in the sense that the curves tend to the equilibrium amplitude solution for the unmodulated flow. Since all the curves lie below Davey's curve we see that modulation stabilizes the flow in the sense that the value of $\bar{A}$ for given values of α and T_1/T_0 is always less than its unmodulated value. However, unlike the results of Donnelly, our results show no optimum value of α and hence σ for a given value of ϵ at which the enhancement of stability is most pronounced. The enhancement of stability shown in figure 5 decreases as α increases. In figure 6 we show the results of our low frequency theory in a form more suitable for comparison with Donnelly's figure 5. We see that there is poor agreement between our

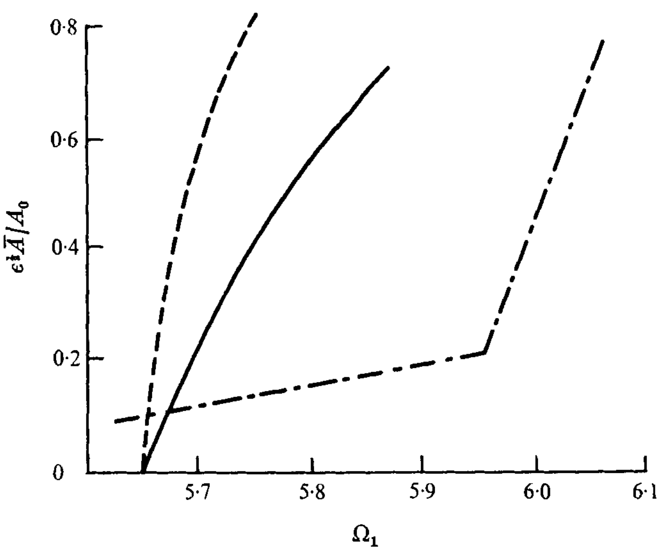

FIGURE 6. Comparison with Donnelly's results for $\epsilon = 0{\cdot}08$ and a period of 46·1. – – –, Davey's equilibrium amplitude solution; — — —, Donnelly's experimental curve; ——, $\bar{A}$ given by low frequency theory. (All amplitudes are normalized by division by A_0, the amplitude at $\Omega = 5{\cdot}8$ without modulation.)

theory and Donnelly's results. This is perhaps due to the fact that $\bar{A}$ may not be the relevant property of $A(\tau)$ as far as his results are concerned.

A more promising method of experimentally checking our low frequency results may be to try and obtain the behaviour of A as a function of τ. This could perhaps be obtained by measuring the difference of the torque per unit length on the inner cylinder from its laminar value, a quantity which is proportional to $A^2(\tau)$. We show A^2 as a function of τ for various values of α and $T_1/T_0 = 0{\cdot}5$ in figure 7.

In the limit in which σ tends to infinity with ϵ arbitrary, we note that the amplitude of the dominant steady fundamental component of the perturbation velocity is given by (7.10). In view of the scaling in (7.4) the physical amplitude A of this component is $\sigma^{-\frac{3}{2}}A_s^0$. If $\epsilon = T_6 = 0$ we can show that A becomes the equilibrium amplitude solution A_E for the problem without modulation at the same Taylor number. We can show that

$$A - A_E = -\Gamma T_6/2a_1(A + A_E)\sigma^3,$$

thus we can see that as the flow becomes more and more supercritical, in which case A and A_E both increase, the difference between them decreases. In figure 8 we have sketched A as a function of T for different values of ϵ.

In contrast to the low frequency results we see that the average amplitude of the Taylor-vortex flow grows quite rapidly as soon as the critical Taylor number of linear theory is reached. Since T_6 is negative, we conclude that in the high frequency limit modulation destabilizes the flow.

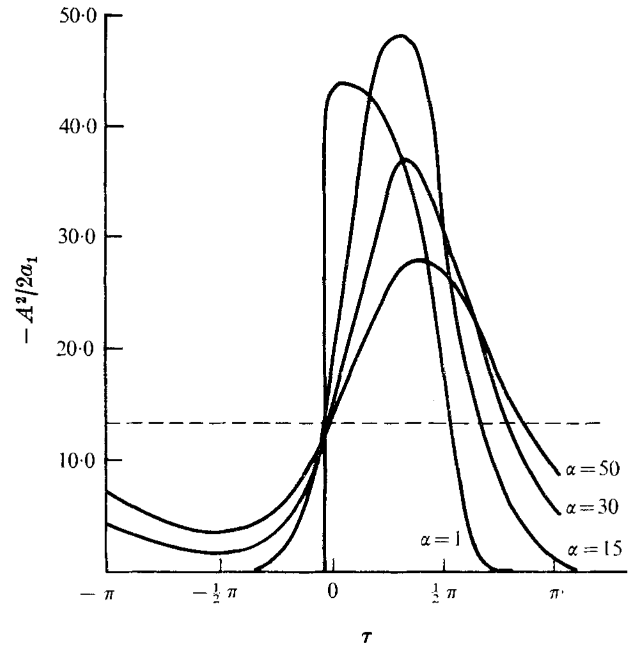

FIGURE 7. A^2 as a function of τ for $T_1/T_0 = 0{\cdot}5$. ---, constant value of A^2 for the unmodulated problem with the same Taylor number. The difference G in the torque on the inner cylinder from its laminar value, for this value of T_1/T_0, can be shown to be given by $G = [2\pi\Omega_1 R_1^3\mu/(R_2-R_1)] \times 0{\cdot}037A^2\epsilon$.

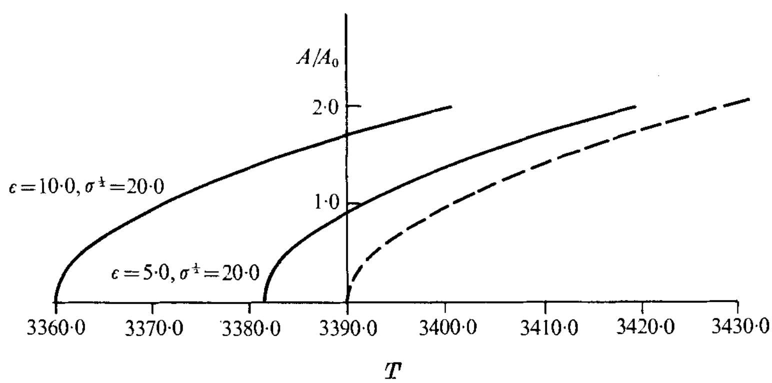

FIGURE 8. The amplitude A as a function of T in the high frequency limit. ---, amplitude for the unmodulated problem for the same Taylor number. (All amplitudes are normalized by division by A_0, the amplitude at $T = 3400$ without modulation.)

Following a suggestion by Professor J. Mahony an alternative approach to this problem has now been formulated for the limits of small ϵ and ϵ/σ. The method is essentially that of Venezian (1969) and we just expand the perturbation velocity in powers of ϵ. By taking the further limits $\sigma \to 0$ and $\sigma \to \infty$ we recover the results of §§3 and 4. At intermediate values of σ we again find that the flow is destabilized. This work can be found in appendix B, which was added to this paper at the proof stage.

The author wishes to acknowledge the receipt of a Science Research Council maintenance grant and the help and guidance given by Professor J. T. Stuart at Imperial College.

Appendix A

Here we consider in more detail some of the functions which appeared in § 4 but were not given explicitly there. We consider first the terms which appear in the expansions of the inner-layer Fourier coefficients u_s, v_s, u_1, v_1, etc. We recall that these functions of η are determined by (4.10)–(4.12). The appropriate boundary conditions are obtained by substituting the expansions (4.7) into (4.6 a) and equating terms proportional to $e^{in\tau}$ for $n = 0, 1, 2, \ldots$. If we substitute the expansions (4.14) into (4.10) we see that the first five terms in the expansions of u_s and v_s can be found by equating terms of order σ^0, $\sigma^{-\frac{1}{2}}$, σ^{-1}, $\sigma^{-\frac{3}{2}}$ and σ^{-2} and solving the resulting differential equations subject to the appropriate boundary conditions. We can then write u_s and v_s in the form

$$\begin{aligned} u_s = \sigma^{-1}\{&B_0\eta^2 + \sigma^{-\frac{1}{2}}[B_1\eta^2 + A_0\eta^3] + \sigma^{-1}[B_2\eta^{-2} + A_1\eta^3 + \tfrac{1}{3}a^2B_0\eta^4] \\ &+ \sigma^{-\frac{3}{2}}[B_3\eta^2 + A_2\eta^3 + \tfrac{1}{3}a^2B_1\eta^4 + \tfrac{1}{5}a^2(A_0 - \tfrac{1}{6}C_0\dot{T}_0)\eta^5] \\ &+ \sigma^{-2}[B_4\eta^2 + A_3\eta^3 + \tfrac{1}{3}a^2B_2\eta^4 + \tfrac{1}{5}a^2(A_1 - \tfrac{1}{6}C_1T_0)\eta^5 \\ &+ \tfrac{1}{30}a^2(a^2B_0 + \tfrac{1}{3}\times 2^{\frac{1}{2}}C_0T_0)\eta^6] + O(\sigma^{-\frac{5}{2}})\}, \end{aligned} \tag{A 1a}$$

$$\begin{aligned} v_s = \sigma^{-\frac{1}{2}}\{&C_0\eta + \sigma^{-\frac{1}{2}}C_1\eta + \sigma^{-1}[C_2\eta + \tfrac{1}{3}a^2C_0\eta^3] \\ &+ \sigma^{-\frac{3}{2}}[C_3\eta + \tfrac{1}{3}a^2C_1\eta^3 + \tfrac{1}{6}B_0\eta^4] + \sigma^{-2}[C_4\eta + \tfrac{1}{3}a^2C_2\eta^3 + \tfrac{1}{6}B_1\eta^4 \\ &+ \tfrac{1}{10}(A_0 + \tfrac{1}{3}a^4C_0)\eta^5] + O(\sigma^{-\frac{5}{2}})\}, \end{aligned} \tag{A 1b}$$

where A_0, B_0, etc., are unknown constants which are to be determined later. For the sake of brevity we have put $T_1 = 0$ in order to evaluate (A 1). The vanishing of T_1 would otherwise be found later. Knowledge of the first five terms in the expansions of u_s and v_s enables us to evaluate the first five terms in the expansions of u_1 and v_1. If we substitute for u_1 and v_1 from (4.14) into (4.11) and equate terms of order σ^0 and $\sigma^{-\frac{1}{2}}$ and use (A 1) we obtain differential equations which when solved subject to the appropriate boundary conditions enable us to write u_1 and v_1 in the form

$$\begin{aligned} u_1 = \sigma^{-\frac{5}{2}}\{&(P_0 + \sigma^{-\frac{1}{2}}P_1)[e^{-\eta(1+i)} - 1 + \eta(1+i)] \\ &- \tfrac{1}{8}a^2\epsilon[C_0T_0 + \sigma^{-\frac{1}{2}}C_1T_0][2(1+i)\eta^2e^{-\eta(1+i)} + 10\eta\, e^{-\eta(1+i)} \\ &+ 5(1-i)(e^{-\eta(1+i)} - 1)] + O(\sigma^{-1})\}, \end{aligned} \tag{A 2a}$$

$$\begin{aligned} v_1 = \sigma^{-\frac{3}{2}}\{&\tfrac{1}{24}\times 2^{\frac{1}{2}}\epsilon(B_0 + \sigma^{-\frac{1}{2}}B_1)e^{-\eta(1+i)}(4\eta^3 + 3\eta^2 - 3i\eta^2 - 3i\eta) \\ &- \tfrac{1}{32}\times 2^{\frac{1}{2}}A_0\epsilon\sigma^{-\frac{1}{2}}[e^{-\eta(1+i)}(4\eta^4 + 4(1-i)\eta^3 - 6i\eta^2 - 3\eta - 3i\eta)] + O(\sigma^{-1})\}, \end{aligned} \tag{A 2b}$$

where P_0 and P_1 are constants to be determined; exponentially increasing functions of η have been rejected. In order to calculate P_0 and P_1 we must consider the flow away from the inner layer.

To this end, we recall that in the central region and the outer layer there is no coupling of the differential equations for the Fourier coefficients in the expansion of the perturbation velocity. We first consider the central region. Suppose that we substitute for U and V from (4.8) into (4.4), equate terms proportional to $e^{i\tau}$ and take U_1 and V_1 as given by (4.16) with $n = 1$. We can equate like powers of $\sigma^{-\frac{1}{2}}$ and solve the resulting differential equations to obtain

$$U_1 = \mu_1\{B_1^0 \sinh a\zeta^* + A_1^0 \cosh a\zeta^* + \sigma^{-\frac{1}{2}}[B_1^1 \sinh a\zeta^* + A_1^1 \cosh a\zeta^*] + O(\sigma^{-\frac{1}{2}})\}, \quad (A\,3a)$$

$$V_1 = (-U_1/i\sigma)\{1 + O(\sigma^{-1})\}, \quad (A\,3b)$$

where A_1^0 and B_1^0, etc., are constants to be determined. A similar procedure in the outer layer shows that

$$u_1^* = \nu_1(\sigma)\{[C_1^0 + \sigma^{-\frac{1}{2}}C_1^1][e^{-\eta^*(1+i)} + (1+i)\eta^* - 1] + O(\sigma^{-1})\}, \quad (A\,4a)$$

$$v_1^* = \frac{\sigma^{-1}\nu(\sigma)}{1+i}\left\{[C_1^0 + \sigma^{-\frac{1}{2}}C_1^1]\left(-\eta^* e^{-\eta^*(1+i)} - \frac{2e^{-\eta^*(1+i)}}{1+i} + \frac{2}{1+i} - 2\eta^*\right) + O(\sigma^{-1})\right\}, \quad (A\,4b)$$

where C_1^0 and C_1^1 are for the moment unknown constants. Again exponentially increasing functions of η^* have been rejected.

It now remains to match (A 2) and (A 3) where the central region and inner layer overlap and (A 3) and (A 4) where the central region and outer layer overlap We can easily show that the conditions in the region of overlap of the outer layer and central region reduce to

$$\mu_1 = \sigma^{\frac{1}{2}}\nu_1,$$

$$A_1^0 = 0, \quad B_1^0 = C_1^0(1+i)/a2^{\frac{1}{2}},$$

$$A_1^1 = C_1^0, \quad B_1^1 = C_1^1(1+i)/a2^{\frac{1}{2}}$$

and the conditions in the other overlap region reduce to

$$\mu_1 = \sigma^{-\frac{3}{2}}, \quad P_0 = 0,$$

$$B_1^0 = 5a^2\epsilon C_0 T_0/4(1+i)\sinh a,$$

$$P_1 = -2^{\frac{1}{2}} \times 5a^3\epsilon C_0 T_0 \coth a/8i,$$

$$B_1^1 = 5a^2\epsilon\left[C_1 T_0 - \frac{8P_1\{1+i\}}{5a^2\epsilon}\right]\Big/4(1+i)\sinh a,$$

$$P_2 = -2^{\frac{1}{2}} \times 5a^2\epsilon\left[C_1 T_0 - \frac{8P_1\{1+i\}}{5a^2\epsilon}\right]\Big/8i\tanh a - \frac{a\,2^{\frac{1}{2}}}{(1+i)}P_1 \tanh a.$$

If it is required to calculate T to higher order it is necessary to calculate u_2, v_2, etc. However, the method follows closely the one described above. If the above expressions for P_0 and P_1 are substituted into (A 2) we can calculate the next two terms in the expansion of u_s and v_s.

$$\begin{aligned}
u_s^5+\sigma^{-\frac{1}{2}}u_s^6 = {} & [B_5+\sigma^{-\frac{1}{2}}B_6]\eta^2+[A_4+\sigma^{-\frac{1}{2}}A_5]\eta^3+\tfrac{1}{3}a^2[B_3+\sigma^{-\frac{1}{2}}B_4]\eta^3 \\
& +\tfrac{1}{5}a^2[A_2+\sigma^{-\frac{1}{2}}A_3-C_2T_0-\sigma^{-\frac{1}{2}}C_3T_0]\eta^5 \\
& +\tfrac{1}{30}a^2\{a^2B_1+\sigma^{-\frac{1}{2}}a^2B_2+\tfrac{1}{3}\times 2^{\frac{1}{2}}(C_1T_0+\sigma^{-\frac{1}{2}}C_2T_0)\}\eta^6 \\
& +\tfrac{1}{70}a^4\{A_0+\sigma^{-\frac{1}{2}}A_1-\tfrac{1}{3}(C_0T_0-\sigma^{-\frac{1}{2}}C_1T_0)\}\eta^7 \\
& \qquad +\tfrac{1}{630}\sigma^{-\frac{1}{2}}a^2B_0\{a^4-\tfrac{1}{4}T_0\}\eta^8 \\
& +\tfrac{1}{192}a^2\epsilon^2(B_0T_0+\sigma^{-\frac{1}{2}}B_1T_0)\,2^{\frac{1}{2}}\{e^{-2\eta}[4\eta^3+27\eta^2+72\eta+75] \\
& +78\eta+75\}+\tfrac{1}{256}a^2\epsilon^2A_0T_0\,2^{\frac{1}{2}}\{e^{-2\eta}[4\eta^4+36\eta^3+144\eta^2 \\
& \qquad +297\eta+264]+231\eta+264\},
\end{aligned}$$

$$\begin{aligned}
v_s^5+\sigma^{-\frac{1}{2}}v_s^6 = {} & [C_5+\sigma^{-\frac{1}{2}}C_6]\eta+\tfrac{1}{3}a^4[C_3+\sigma^{-\frac{1}{2}}C_4]\eta^3+\tfrac{1}{6}[B_2+\sigma^{-\frac{1}{3}}B_3]\eta^4 \\
& +\tfrac{1}{10}[A_1+\tfrac{1}{3}a^4C_1+\sigma^{-\frac{1}{2}}(A_2+\tfrac{1}{3}a^4C_2)]\eta^5+\tfrac{1}{30}a^2[B_0+\sigma^{-\frac{1}{2}}B_1]\eta^6 \\
& -\tfrac{1}{32}a^2\epsilon^2[C_0T_0+\sigma^{-\frac{1}{2}}C_1T_0]\,2^{\frac{1}{2}}\{e^{-2\eta}[2\eta^2+9\eta+8]-10\cos\eta\, e^{-\eta}+2\} \\
& +\sigma^{-\frac{1}{2}}\{\tfrac{1}{70}a^2[A_0+\tfrac{1}{9}a^4C_0-\tfrac{1}{9}C_0T_0]\eta^7+\tfrac{5}{32}a^3\epsilon^2C_0T_0\coth a[e^{-2\eta} \\
& \qquad -2e^{-\eta}(3\cos\eta-\sin\eta)-4\eta\, e^{-\eta}\cos\eta+1]\},
\end{aligned}$$

where A_4, etc., are unknown constants and for convenience we have put $T_2 = T_3 = 0$. The vanishing of these quantities would otherwise be found from the matching conditions. If we calculate a few more terms in the expansions of u_s and v_s it is easy to deduce the matching condition (4.21).

Appendix B

We now describe briefly an alternative formulation of the problem in the limit $\epsilon \to 0$. Suppose that the disturbance (u, v, w) imposed on the flow is small enough for linearization to be a valid procedure. If the disturbance is periodic along the axis of the cylinders with wavenumber a then the appropriate form of (2.3) is

$$\left.\begin{aligned}
\left\{\frac{\partial^2}{\partial\zeta^2}-a^2-\sigma\frac{\partial}{\partial\tau}\right\}\left\{\frac{\partial^2}{\partial\zeta^2}-a^2\right\}u &= -a^2Tv\left\{\chi_0+\frac{\epsilon e^{i\tau}}{2}f(\zeta,\sigma)+\frac{\epsilon e^{-i\tau}}{2}\tilde{f}(\zeta,\sigma)\right\}, \\
\left\{\frac{\partial^2}{\partial\zeta^2}-a^2-\sigma\frac{\partial}{\partial\tau}\right\}v &= -u\left\{-1+\frac{\epsilon e^{i\tau}}{2}\frac{df}{\partial\zeta}(\zeta,\sigma)+\frac{\epsilon e^{-i\tau}}{2}\frac{d\tilde{f}}{d\zeta}(\zeta,\sigma)\right\}, \\
u = v = \partial u/\partial\zeta &= 0, \quad \zeta = 0, 1,
\end{aligned}\right\} \quad \text{(B 1)}$$

where $$f = \sinh(i\sigma)^{\frac{1}{2}}(1-\zeta)/\sinh(i\sigma)^{\frac{1}{2}},$$

and the boundary conditions shown stipulate that there is no fluid motion relative to the cylinders, which, in terms of ζ, correspond to $\zeta = 0$ and $\zeta = 1$.

Following Venezian (1969) we expand u, v and T in the form

$$u = u_0+\epsilon u_1+\epsilon^2u_2+\dots, \quad \text{(B 2a)}$$

$$v = v_0+\epsilon v_1+\epsilon^2v_2+\dots, \quad \text{(B 2b)}$$

$$T = T_0+\epsilon^2T_2+\epsilon^4T_4+\dots, \quad \text{(B 2c)}$$

where we have anticipated that $T_i = 0$ for i odd. This is because changing ϵ to $-\epsilon$ does not alter the physical problem under investigation. We can substitute from (B 2) into (B 1) and equate terms of order ϵ^0 and ϵ respectively to show that

$$(u_0, v_0) = A(f_0, g_0), \quad (u_1, v_1) = \tfrac{1}{2}A\, e^{i\tau}(f_{11}, g_{11}) + \tfrac{1}{2}Ae^{-i\tau}(\tilde{f}_{11}, \tilde{g}_{11}), \qquad \text{(B 3}a, b)$$

where (f_0, g_0) and (f_{11}, g_{11}) are determined by

$$\left.\begin{aligned} N^2 f_0 + a^2 T_0 \chi_0 g_0 = f_0 - N g_0 = 0,\\ f_0 = df_0/d\zeta = g_0 = 0, \quad \zeta = 0, 1,\end{aligned}\right\} \qquad \text{(B 4)}$$

$$\left.\begin{aligned} N(N-i\sigma) f_{11} + a^2 T_0 \chi_0 g_{11} &= -a^2 T_0 g_0 f,\\ (N-i\sigma)\, g_{11} - f_{11} &= -f_0\, df/d\zeta,\\ f_{11} = df_{11}/d\zeta = g_{11} &= 0, \quad \zeta = 0, 1.\end{aligned}\right\} \qquad \text{(B 5)}$$

The operator N is defined by

$$N \equiv d^2/d\zeta^2 - a^2,$$

and A is a constant which cannot be determined within the framework of linear theory. At order ϵ^2 we find that (u_2, v_2) is expressible in the form

$$(u_2, v_2) = \tfrac{1}{2}Ae^{2i\tau}(f_{22}, g_{22}) + \tfrac{1}{2}Ae^{-2i\tau}(\tilde{f}_{22}, \tilde{g}_{22}) + (f_{20}, g_{20}), \qquad \text{(B 6)}$$

and the integral condition that the differential system determining (f_{20}, g_{20}) has a solution gives

$$a^2 T_2 = \int_0^1 \left(-f_0^+\{\tilde{g}_{11} f + g_{11}\tilde{f}\} + g_0^+\left\{f_{11}\frac{d\tilde{f}}{d\zeta} + \tilde{f}_{11}\frac{df}{d\zeta}\right\}\right) d\zeta \Big/ A \int_0^1 \chi_0 f_0^+ g_0\, d\zeta. \quad \text{(B 7)}$$

Here (f_0^+, g_0^+) is the function pair adjoint to (f_0, g_0) and is defined by

$$\left.\begin{aligned} N^2 f_0^+ + g_0^+ = N g_0^+ - a^2 T_0 \chi_0 f_0^+ = 0,\\ f_0^+ - df_0^+/d\zeta = g_0^+ = 0, \quad \zeta = 0, 1.\end{aligned}\right\} \qquad \text{(B 8)}$$

The systems (B 4) and (B 8) are independent of σ and therefore need only be integrated once. However, (B 5) and (B 7) depend on σ and so must be integrated separately for each value of σ. The results of such a procedure are given later. We now show how the results of §§ 3 and 4 can be recovered by taking the further limits $\sigma \to 0$ and $\sigma \to \infty$.

The further limits $\sigma \to 0$ and $\sigma \to \infty$

In § 3 we saw that in the limit $\epsilon \to 0$ with σ/ϵ fixed we can express T in the form

$$T = T_0 + \epsilon^2 T_2^* + \epsilon^2\sigma^2 T_4^* + O(\epsilon^2\sigma^4, \epsilon^4), \qquad \text{(B 9)}$$

and when T_0 has its critical value, 3390, we have that $T_2^* = -208{\cdot}6$ and $T_4^* = 1{\cdot}7$.

Suppose that we expand (f_{11}, g_{11}) in the form

$$(f_{11}, g_{11}) = \sigma^{-1}(u_{11}^0, v_{11}^0) + (u_{11}^1, v_{11}^1) + \sigma(u_{11}^2, v_{11}^2) + \ldots, \qquad \text{(B 10)}$$

then by substituting this expansion into (B 5) with f expanded for small σ, we can equate terms of order σ^{-1}, σ^0 and σ respectively to show that

$$(f_{11}, g_{11}) = i\sigma^{-1}(\Gamma f_0, \Gamma g_0) + (f_1 + \Gamma_1 f_0, g_1 + \Gamma_i g_0) + O(\sigma). \qquad \text{(B 11)}$$

Here (f_1, g_1) and Γ are as defined by (3.14) and (3.12) and Γ_1 is defined by equation (3.3.19*a*) of Hall (1973). If we take (f_{11}, g_{11}) as shown above and expand f for small σ in (B 7) we can show that

$$a^2T_2 = \frac{-\frac{1}{2}\int_0^1 \{f_0^+[\Gamma N f_1 + a^2T_0\chi_0 g_1] + g_0^+[f_1 - \Gamma g_1]\}\,d\zeta + O(\sigma^2)}{\int_0^1 f_0^+\chi_0 g_0\,d\zeta} \tag{B 12}$$

and the dominant term above is just the term $a^2T_2^*$ of § 3. Similarly the order-σ^2 term in (B 12) can be shown to be the $a^2T_4^*$ of § 3. Thus we can recover the results of § 3 by taking the further limit $\sigma \to 0$. However, the velocity field obtained above is not identical to that found in § 3. In order to see why this is so we first note that the first-order correction to the unmodulated Taylor-vortex flow obtained above is of order ϵ/σ, equal to α^{-1} say. Thus our analysis is restricted to $\alpha \gg 1$ in contrast to § 3, where we had $\sigma \sim \epsilon$. However the results for $\alpha \gg 1$ can be found from § 3 by taking the limit $\alpha \to \infty$; the resulting velocity field is then identical to that derived above. Thus the work of this section is just a special case of § 3.

When $\sigma \to \infty$ we found in § 4 that T could be written in the form

$$T = T_0 + \sigma^{-3}T_6 + \sigma^{-\frac{7}{2}}T_7 + \ldots, \tag{B 13}$$

and when $T_0 = 3389{\cdot}9$ we found that

$$T_6/T_0 = -4{\cdot}898(10^4\times\epsilon^2) \quad\text{and}\quad T_7/T_0 = -84{\cdot}81(10^4\times\epsilon^2).$$

When $\sigma \to \infty$ we have that $f \sim \exp[-(i\sigma)^{\frac{1}{2}}]$, so that the time dependence of the basic flow is confined to a thin Stokes layer at the inner cylinder. We refer to this layer as the inner layer and define a stretched variable for this layer by

$$\eta = \zeta(\tfrac{1}{2}\sigma)^{\frac{1}{2}}, \tag{B 14}$$

and note from (B 4) and (B 7) that in this layer

$$\left.\begin{aligned} f_0 &\sim \tfrac{1}{2}f_0''(0)\,\zeta^2 \sim \sigma^{-1}f''(0)\,\eta^2, & g_0 &\sim g_0'(0)\,\zeta \sim (\tfrac{1}{2}\sigma)^{-\frac{1}{2}}g_0'(0)\,\eta, \\ f_0^+ &\sim \tfrac{1}{2}f_0^{+\prime\prime}(0)\,\zeta^2 \sim \sigma^{-1}f_0^{+\prime\prime}(0)\,\eta^2, & g_0^+ &\sim g_0^{+\prime}(0)\,\zeta \sim (\tfrac{1}{2}\sigma)^{-\frac{1}{2}}g_0^{+\prime}(0)\,\eta, \end{aligned}\right\} \tag{B 15}$$

where a prime denotes a derivative with respect to ζ. If we rescale (B 5) in terms of η and use (B 15) we see that in the inner layer (f_{11}, g_{11}) is determined by

$$\left.\begin{aligned} \frac{d^2}{d\eta^2}\left\{\frac{d^2}{d\eta^2} - 2i\right\}f_{11} &= -\left(\frac{2}{\sigma}\right)^{\frac{5}{2}} aT_0 g_0'(0)\,\eta e^{-\eta(1+i)} + O(\sigma^{-3}), \\ \{d^2/d\eta^2 - 2i\}\,g_{11} &= (2/\sigma^3)^{\frac{1}{2}} f_0''(0)\,(1+i)\,\eta^2 e^{-\eta(1+i)} + O(\sigma^{-2}), \\ f_{11} = g_{11} &= df_{11}/d\eta = 0, \quad \eta = 0. \end{aligned}\right\} \tag{B 16}$$

Thus we can immediately see that in this layer $f_{11} \sim \sigma^{-\frac{5}{2}}$ and $g_{11} \sim \sigma^{-\frac{3}{2}}$, which is consistent with the orders of magnitude of the corresponding dominant unsteady velocity components (proportional to $e^{i\tau}$) found in § 4. Since when $\sigma \to \infty$ the function $f(\zeta, \sigma)$ is exponentially small away from the inner layer we need only evaluate (f_{11}, g_{11}) there to evaluate (B 5). Using (B 15) and (B 16) we immediately see that $T_2 \sim \sigma^{-3}$, which is again consistent with the results of § 4. Unfortunately the solution of (B 16) contains an unknown constant which can only be determined by matching onto the corresponding solution away from the inner layer. Thus if we want to determine the coefficient of this order-σ^{-3} term we must follow the method of § 4 and use the method of matched asymptotic expansions. It suffices here to say that such a procedure shows that f_{11} and g_{11} are just the terms $\sigma^{-\frac{5}{2}}u_1^0$ and $\sigma^{-\frac{3}{2}}v_1^0$ of § 4 respectively. Using the expressions for u_1^0 and v_1^0 found in § 4 and (B 15) we can show that

$$\frac{T_2}{T_0} = \frac{13 f_0^{+\prime\prime}(0) f_0''(0) - 4 g_0^{+\prime}(0)\, g_0'(0)}{\sigma^3\, 32 \int_0^1 \chi_0 f_0^+ g_0 \, d\zeta} + O(\sigma^{-\frac{7}{2}}). \tag{B 17}$$

If we substitute for T_2 from (B 17) into (B 2) we find that the asymptotic form obtained is identical up to order $\epsilon^2\sigma^{-3}$ with that found in § 4. Similarly, by evaluating the order-$\sigma^{-\frac{7}{2}}$ term in (B 17) we find that the expansions are identical to order $\epsilon^2\sigma^{-\frac{7}{2}}$.

Thus we see that by taking the further limit $\sigma \to \infty$ we can recover the results of § 4. The constraint that $\epsilon \ll 1$ makes it easier to see of what order in σ the first correction term of T from T_0 should be in the high frequency limit. However, if we wish to evaluate this term precisely, we must perform the matched asymptotic analysis given in § 4, which we recall was valid for arbitrary ϵ.

Results and discussion

As a starting point for the numerical calculations we assumed the following critical values for a and T_0:

$$a = 3{\cdot}1266, \quad T_0 = 3389{\cdot}9. \tag{B 18}$$

Using the above values we first solved (B 4) and (B 8) by a fourth-order Runge–Kutta scheme with 140 steps. For a given value of σ we then solved (B 5) by a similar procedure and used the results to integrate using Simpson's rule. The results are shown in figure 9, where we have plotted T_2 as a function of σ. We see that T_2 increases monotonically to zero as σ increases. The maximum destabilization is in the limit of zero frequency.

In figure 10 we have compared the numerical solution with the low frequency asymptotic solution of § 3. The results agree well up to $\sigma \sim 4$, where the numerical solution rises less steeply than the asymptotic solution. In figure 11 we have made a similar comparison in the high frequency limit. We see that the numerical and asymptotic solutions agree well for $\sigma \sim 700$. Below this value of σ the solutions diverge slowly, the asymptotic solution eventually crossing the T_2 axis.

We now discuss some possible reasons for the discrepancy between the experimental and theoretical results. It is possible that by working to higher order in ϵ the discrepancy would be removed. However there are reasons why we believe this to be very unlikely. First, the values of ϵ used by Donnelly were typically of order 0·1, so that we can confidently expect effects at orders ϵ^4, ϵ^6, etc., to be negligible. The maximum enhancement of stability observed by Donnelly

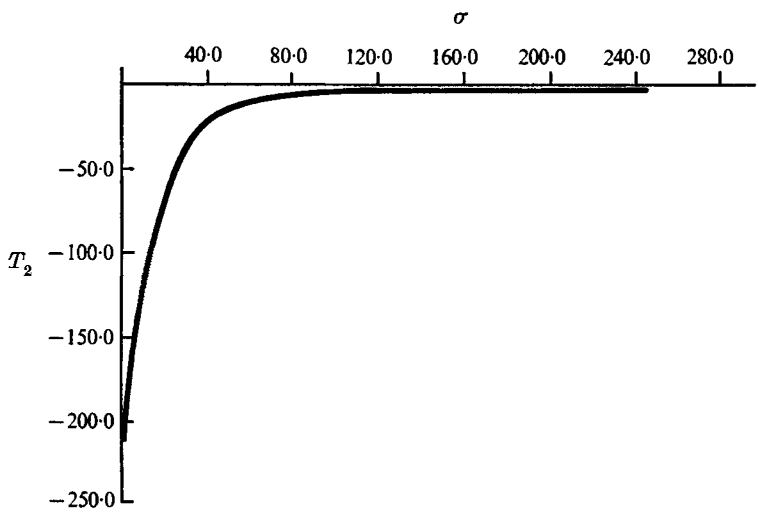

FIGURE 9. Order-ϵ^2 correction to T from T_0 as a function of σ.

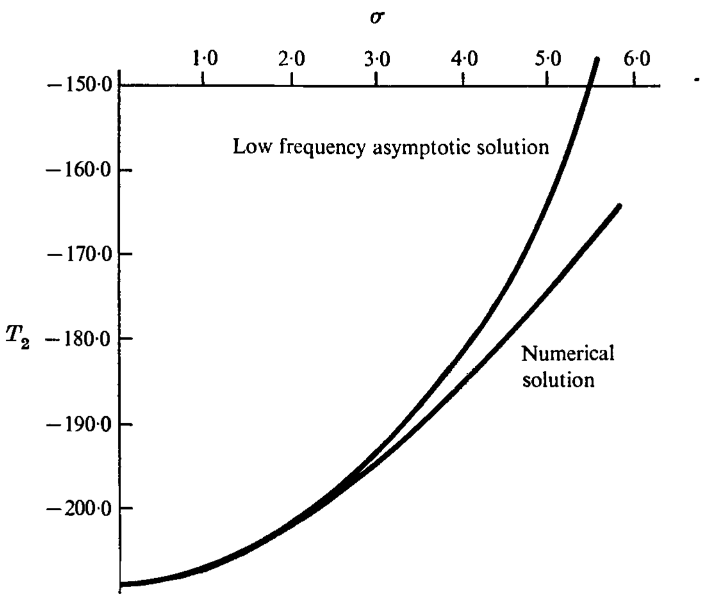

FIGURE 10. Comparison of the low frequency asymptotic solution and the numerical solution for small σ.

was for $\sigma \sim 0{\cdot}2$. We have already seen that the order-ϵ^2 correction to T from T_0 obtained numerically agrees well with the low frequency results of §3 for σ of this order of magnitude. We might also expect that a similar agreement would be found for the order-ϵ^4 term in the expansion of T. However, the method of §3 shows that for small σ this term, which we denote by T_4, is expressible in the form

$$T_4 = T_{40} + \sigma^2 T_{42} + \sigma^4 T_{44} \ldots, \tag{B 19}$$

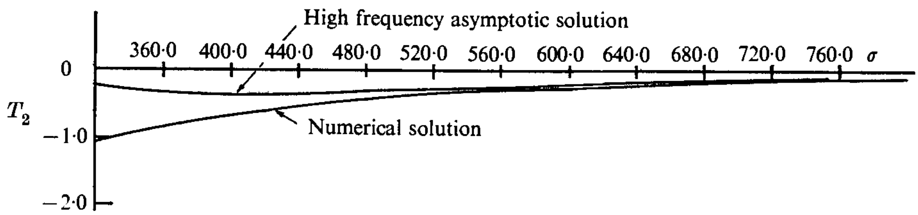

FIGURE 11. Comparison of the high frequency asymptotic solution and the numerical solution for large σ.

and T_{40} was found by Hall (1973) to have the numerical value -1303. (This value includes a correction due to the fact that the critical value of a is altered by an amount of order ϵ^2 from its unmodulated value.) Thus, unless the higher-order terms in (B19) are extremely large and positive, there is no possibility of T_4 producing the pronounced stabilization observed by Donnelly. Moreover, since Donnelly's optimum value of σ was independent of ϵ, such an effect cannot be explained by summing contributions of orders ϵ^2, ϵ^4, etc., since any optimum value of σ obtained by such a procedure would necessarily depend on ϵ.

Finally we should like to say a few words about how nonlinear effects can be put into the framework of this appendix. Suppose that the Taylor number is perturbed by an amount of order ϵ^2 from its unmodulated value T_0. Thus we write

$$T = T_0 + \epsilon^2 T_2^+ + \dots, \tag{B20}$$

and the flow is therefore stable or unstable according to the linear theory of §2 depending on whether T_2^+ is less or greater than T_2, defined by (B7). We know from the unmodulated problem that when the Taylor number T is slightly greater than its critical value T_0 the amplitude of the Taylor-vortex velocity field is then proportional to $(T-T_0)^{\frac{1}{2}}$. Hence, in view of (B20), we expect that any equilibrium perturbation to the modulated flow will have amplitude of order ϵ. Thus we expand the perturbation velocity in the form

$$\begin{aligned} u = {} & A\epsilon u_0 \cos az + \epsilon^2\{(u_1 e^{i\tau} + \tilde{u}_1 e^{-i\tau})\cos az + u_{22}\cos 2az + u_{20}\} \\ & + \epsilon^3\{u_{31}\cos az + u_{33}\cos 3az + (u_{13}e^{i\tau} + \tilde{u}_{13}e^{-i\tau})\cos az \\ & + (u_{32}e^{i\tau} + \tilde{u}_{32}e^{-i\tau})\cos 2az + (u_{30}e^{i\tau} + \tilde{u}_{30}e^{-i\tau}) \\ & + (u_{34}e^{2i\tau} + \tilde{u}_{34}e^{-2i\tau})\cos az\} + O(\epsilon^4), \quad \text{etc.} \end{aligned} \tag{B21}$$

All the terms apart from the first in (B21) are produced by nonlinear interactions. However, some of these interactions are between the basic flow and the disturbance whilst some involve only the disturbance. The terms u_0, u_{22} and u_{20} are the usual fundamental, mean and first-harmonic terms for the unmodulated problem. These terms are produced by interactions involving only the disturbance. (In the notation of Davey (1962) we have $u_0 = \bar{u}_1$, $u_{20} = 0$ and $u_{22} = -\frac{1}{4}\bar{u}_2$.) The term u_1 arises from the interaction of the basic flow and the fundamental and is just the term f_{11} introduced in §2. The order-ϵ^3 terms are produced by interactions between the order-ϵ^2 terms and either the fundamental or the unsteady part of the basic flow. The term u_{31} has contributions from both such interactions. Without modulation we find that the solvability condition on the differential system determining the order-ϵ^3 steady fundamental term is

$$A^2 = \Gamma T_2^+/2a_1 T_0, \tag{B22}$$

where Γ and a_1 are negative constants determined by integral conditions involving the steady fundamental, mean and first-harmonic terms. The value of A determined by (B22) is just the equilibrium amplitude solution of Davey's (1962) truncated third-order amplitude equation. With modulation we find that (B22) becomes

$$A^2 = \left(\frac{\Gamma}{2a_1 T_0}\right)\{T_2^+ - T_2\}, \tag{B23}$$

where T_2 is defined by (B7). Thus finite amplitude perturbations to the flow can exist only when T is greater than its critical value.

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